SAB2223 Mechanics of Materials and Structures

TOPIC 4
STRESS TRANSFORMATION

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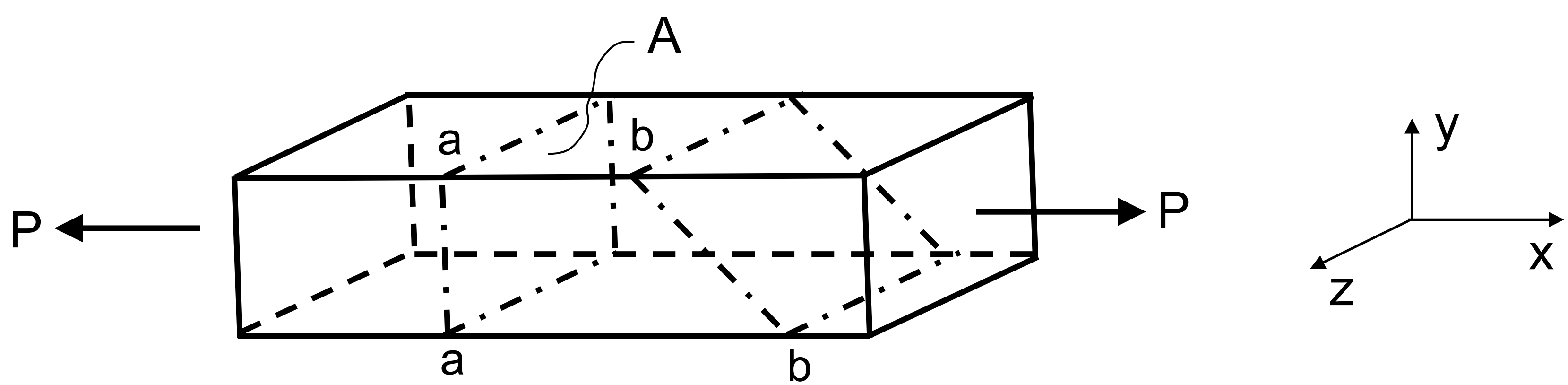
STRESS TRANSFORMATION

Analysis of Stress

- For this topic, the stresses to be considered are not on the perpendicular and parallel planes only but also on other inclined planes.



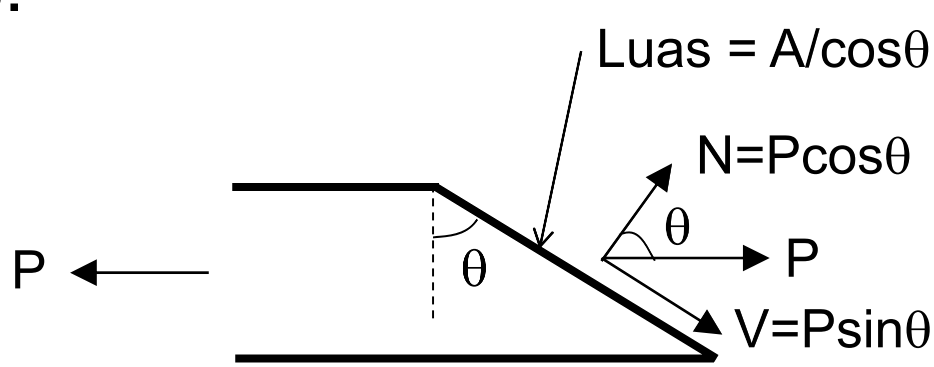
(A body subjected to load P)

Analysis of Stress

- On plane a-a, normal force, N produces normal stress.

$$\sigma = \frac{N}{A} = \frac{P}{A}$$

- On plane b-b, normal force, N produces normal stress and shear force, V produces shearing stress.



Analysis of Stress

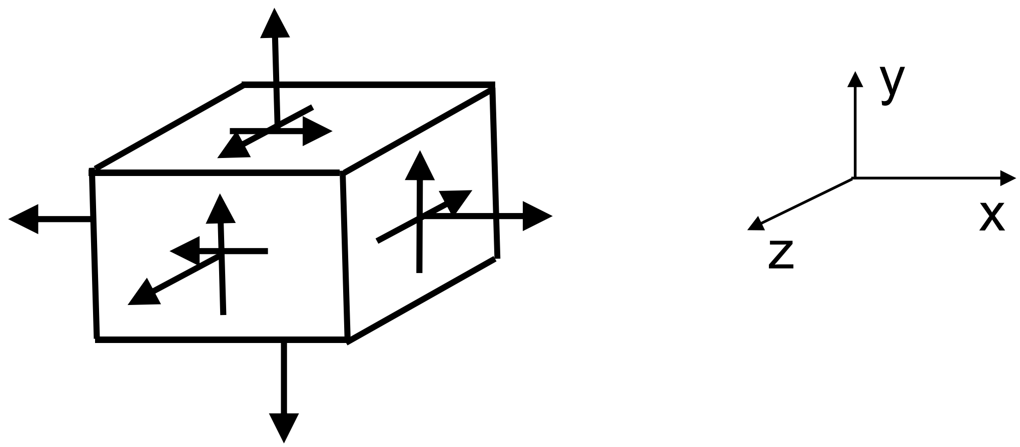
$$\sigma_N = \frac{P\cos\theta}{\left(\dfrac{A}{\cos\theta}\right)} = \frac{P}{A}\cos^2\theta = \sigma_x \cos^2\theta$$

$$\tau = \frac{P\sin\theta}{\left(\dfrac{A}{\cos\theta}\right)} = \frac{P}{A}\sin\theta\cos\theta = \frac{1}{2}\sigma_x \sin 2\theta$$

Analysis of Stress

- On an element, there are 6 components of stress :

$$\sigma_x , \sigma_y , \sigma_z , \tau_{xy} , \tau_{yz} , \text{and } \tau_{zx}$$

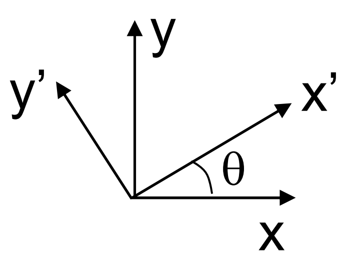


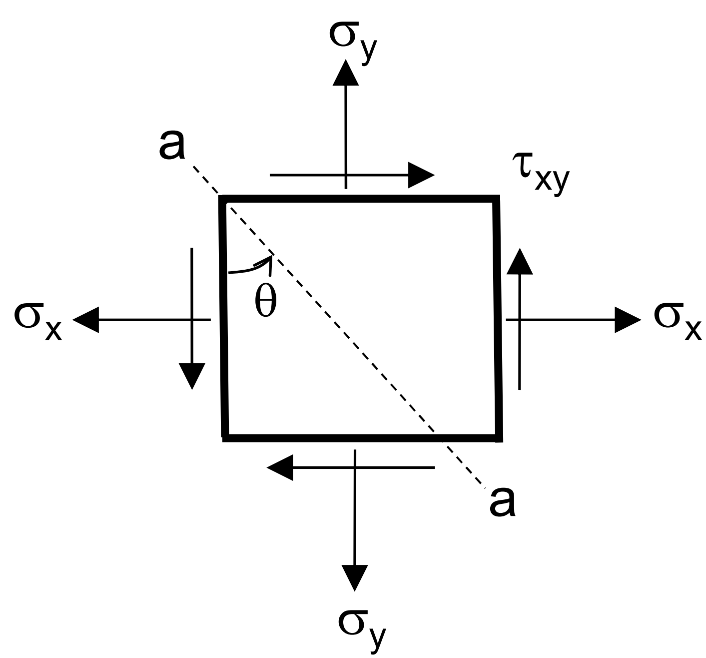
- If the *z* axis is not considered or the stresses are independent with the *z* axis, then there exist only stresses in the *x* and *y* directions.

Analysis of Stress

- This state of 2 dimensional stress is known as plane stress.

$$\sigma_z = \tau_{zx} = \tau_{yz} = 0$$





All directions shown (axes and on element) are taken as positive.

Stress Transformation

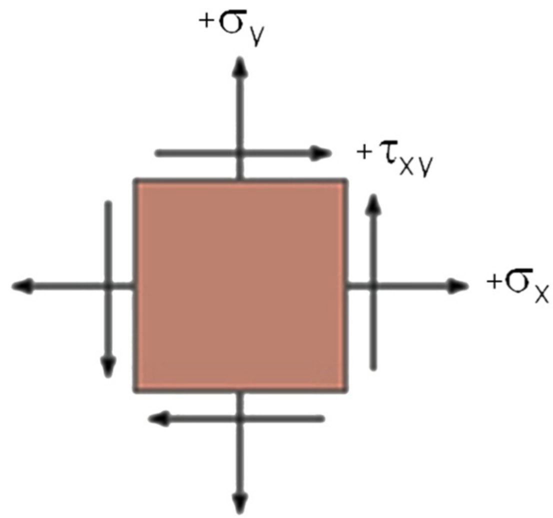
- Stresses on an element can be transformed using 2 methods:
  i) Equations
  ii) Mohr circle

i) EQUATIONS

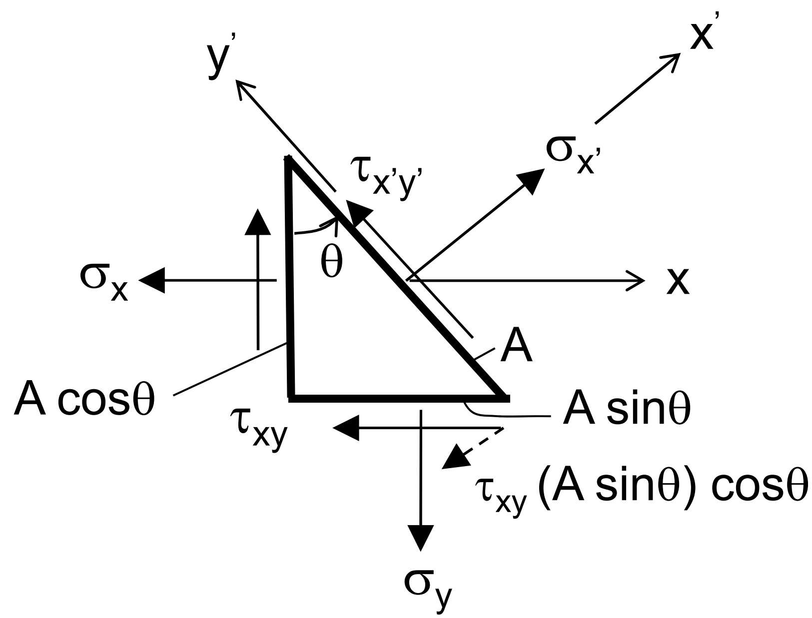
- Consider an element rotated an amount of θ about the z axis.

Stress Transformation

- Stresses on an inclined plane θ will be yielded and can be expressed in term of σ_x , σ_y , τ_{xy} and θ.



Sign Convention:



θ Positive if counter-clockwise and usually taken from the vertical surface (x plane) to the intended plane

σ Positive if tension or in the direction of positive axis

τ Positive if in the direction of positive axis

Stress Transformation

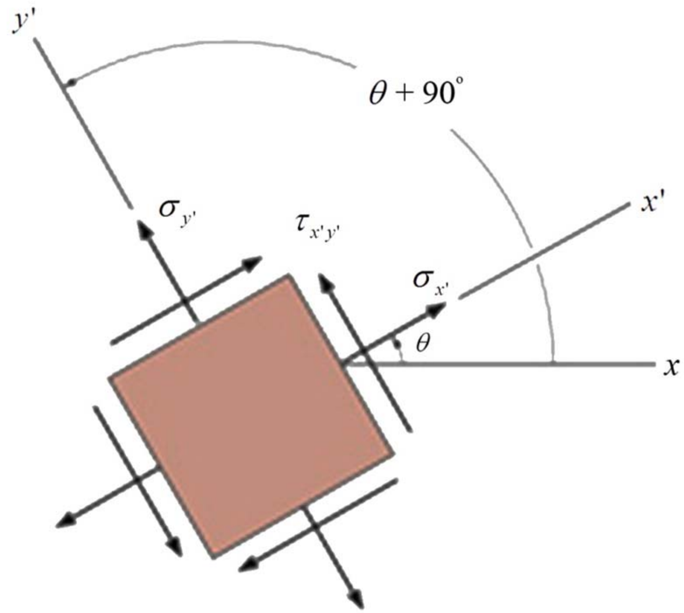
Normal Stress:

$$\sigma_{x'} = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{y'} = \frac{(\sigma_x + \sigma_y)}{2} - \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

Shear Stress:

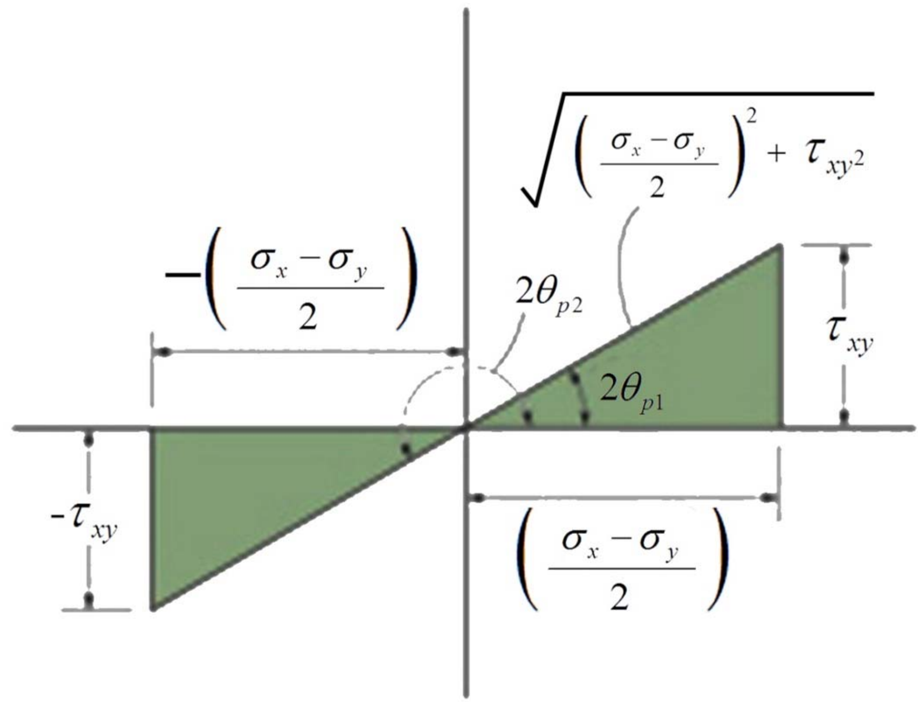
$$\tau_{x'y'} = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$



Stress Transformation

In-Plane Principle Stress:

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\left(\sigma_x - \sigma_y\right)}$$



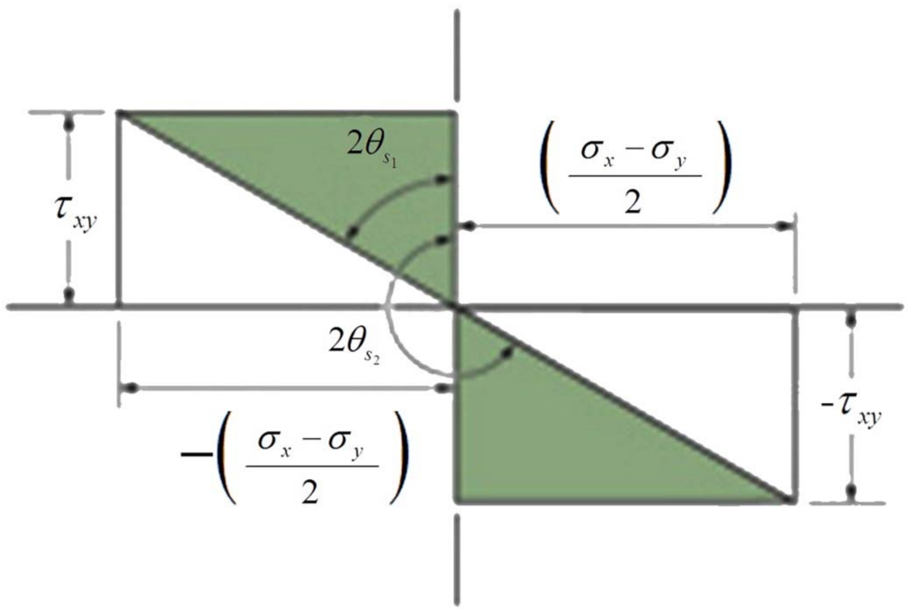
$$\sigma_{\max} \ and \ \sigma_{\min} = \frac{\left(\sigma_x + \sigma_y\right)}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

Stress Transformation

Maximum In-Plane Shear Stress:

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

$$\tau_{max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$



Stress Transformation

Average Normal Stress:

When substituting the values for $2\theta_s$ into the equation for normal stress ($\sigma_{x'}$), there is also a normal stress on the plane of maximum in-plane shear stress, which can be determined by:

$$\sigma_{avg} = \frac{(\sigma_x + \sigma_y)}{2}$$

Stress Transformation

ii) Mohr circle

- In this method, stresses on a plane are drawn as one point on a Mohr circle.

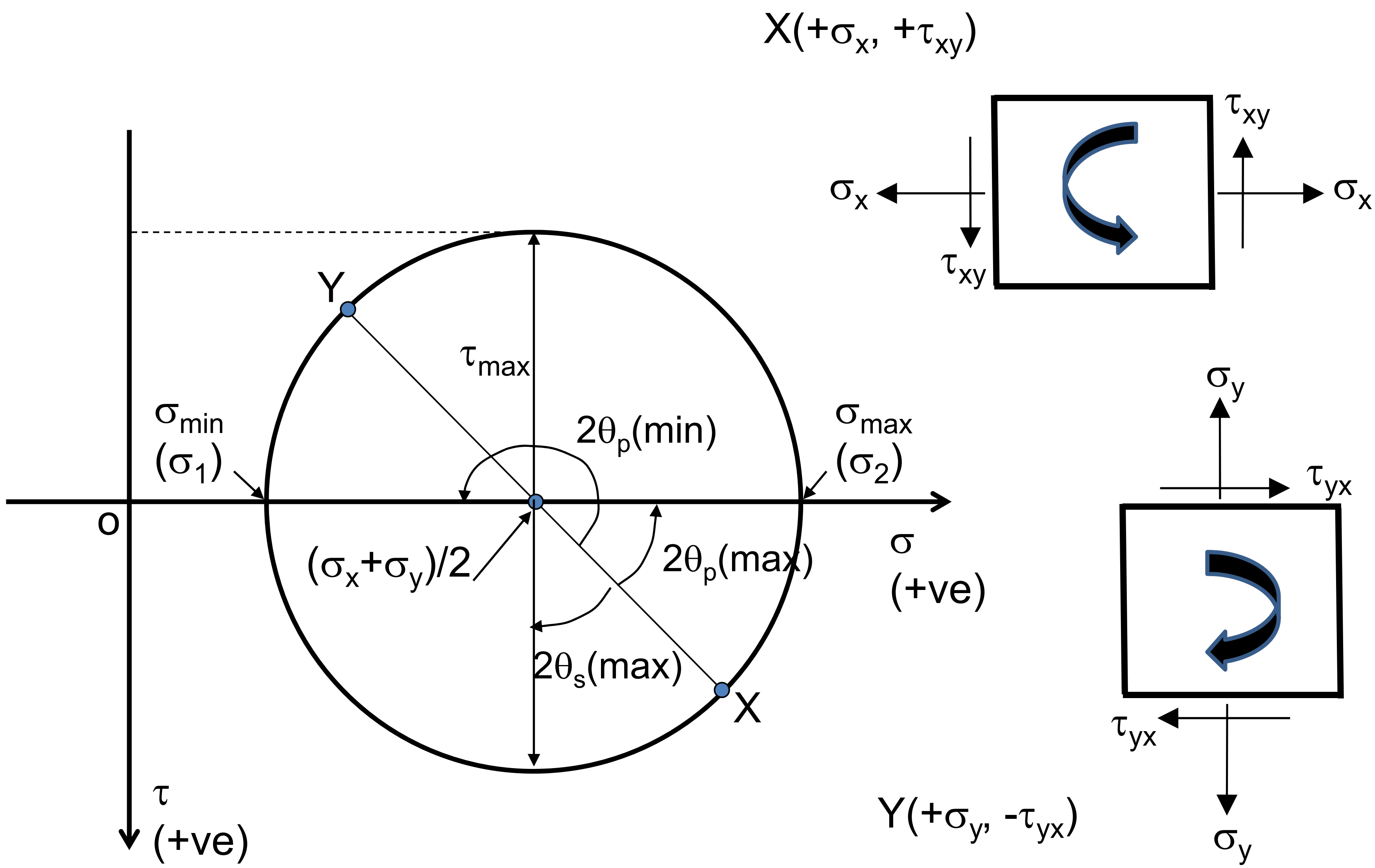
- From the equations, it can be shown that:

$$\underbrace{\left[\sigma_{x'} - \left(\frac{\sigma_x + \sigma_y}{2}\right)\right]^2}_{\text{Centre of Circle}} + \left[\tau_{x'y'}\right]^2 = \underbrace{\left[\frac{\left(\sigma_x - \sigma_y\right)}{2}\right]^2 + \left[\tau_{xy}\right]^2}_{R^2 = \tau_{max}^2}$$

Centre of Circle

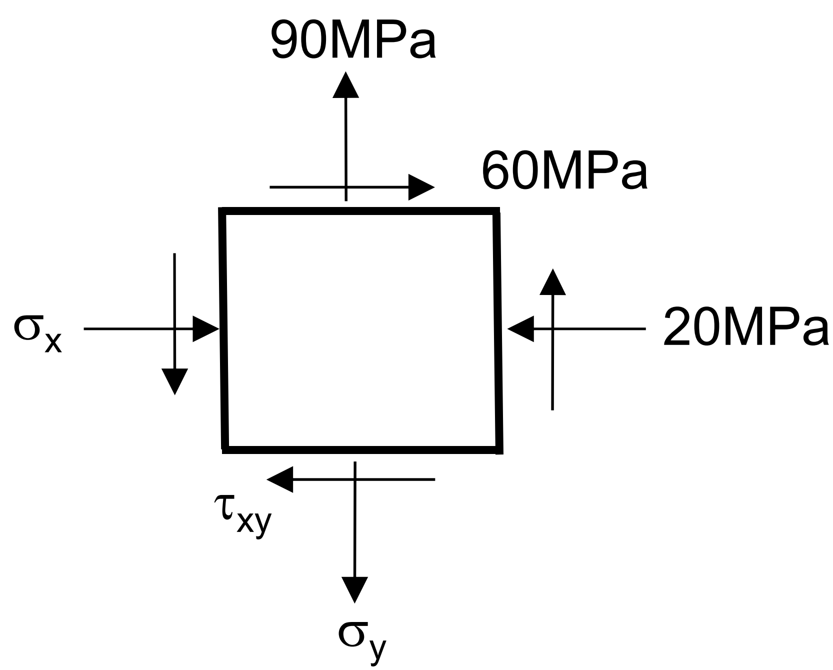
$R^2 = \tau_{max}^2$

Stress Transformation



Example 1

The state of plane stress at a point on a body is shown on the element in the Figure. Represent this stress state in terms of the principal stresses.



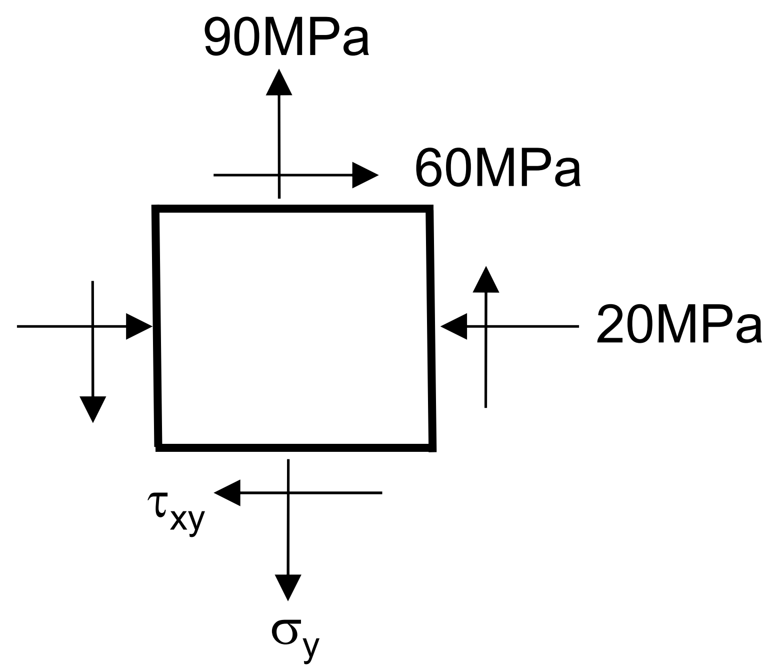
Example 1 (cont.)

Solution

$\sigma_x = -20MPa$ (Compression)

$\sigma_y = 90MPa$ (Tension)

$\tau_{xy} = 60MPa$ (Clockwise)



Example 1 (cont.)

Principal Stresses

$$\sigma_{\max}\ and\ \sigma_{\min} = \frac{(\sigma_x + \sigma_y)}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_{\max}\ and\ \sigma_{\min} = \frac{(-20 + 90)}{2} \pm \sqrt{\left(\frac{-20 - 90}{2}\right)^2 + 60^2}$$

$$\sigma_{\max}\ and\ \sigma_{\min} = 35 \pm 81.4$$

$$\sigma_{\max} = 35 + 81.4 = 116 MPa$$

$$\sigma_{\min} = 35 - 81.4 = -46.4 MPa$$

Example 1 (cont.)

Orientation of Element

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

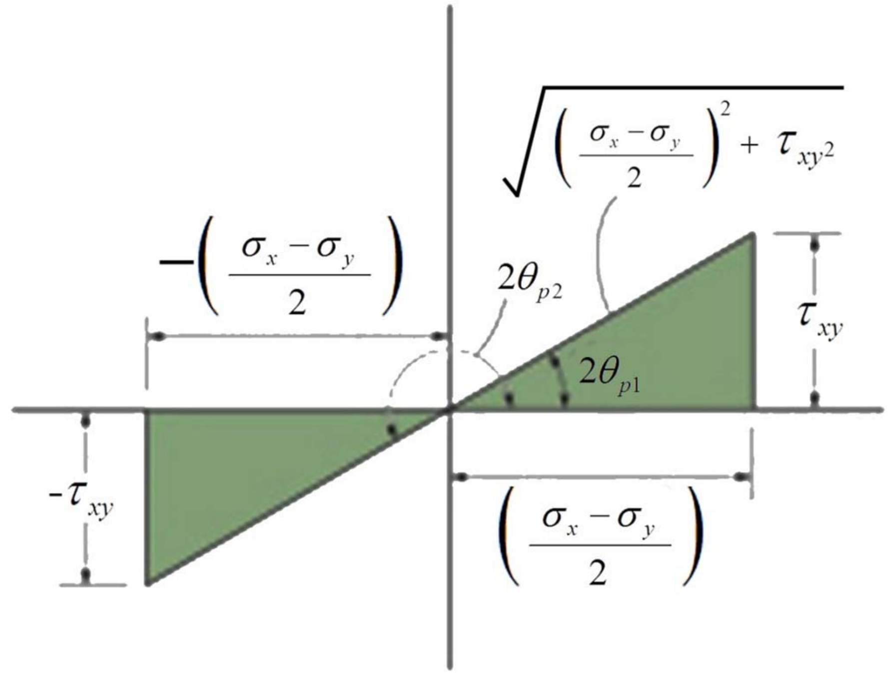
$$\tan 2\theta_p = \frac{2 \times 60}{(-20 - 90)}$$

$$2\theta_{p2} = -47.49^\circ$$

$$\theta_{p2} = -23.7^\circ$$

$$2\theta_{p1} = -47.49^\circ + 180^\circ = 132.51^\circ$$

$$\theta_{p1} = 66.3^\circ$$



Example 1 (cont.)

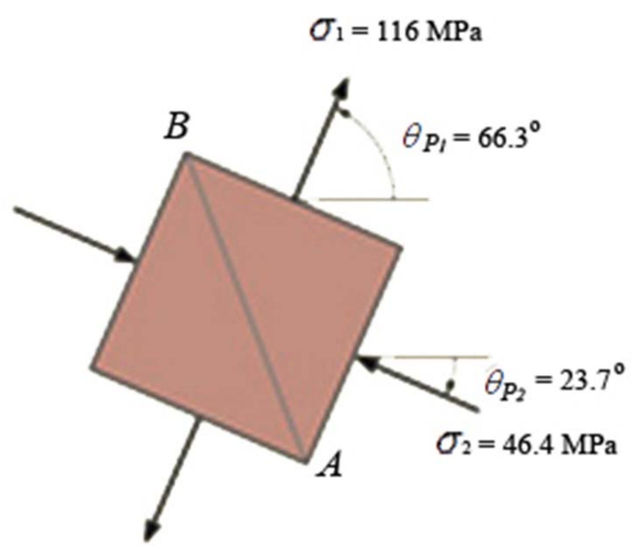
The principal plane on which each normal stress acts can be determined by applying:

$$\sigma_{x'} = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{x'} = \frac{(-20+90)}{2} + \frac{(-20-90)}{2}\cos(-47.49°) + 60\sin(-47.49°)$$

$$\sigma_{x'} = -46.4MPa$$

Hence, σ_{min} = -46.4MPa acts on the plane defined by θ_{p2} = -23.7°, whereas σ_{max} = 116MPa acts on the plane defined by θ_{p1} = 66.3°.



Example 1 (cont.)

By replacing the θ_{p1} and θ_{p2} into the equation for shear stress:

$$\tau_{x'y'} = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

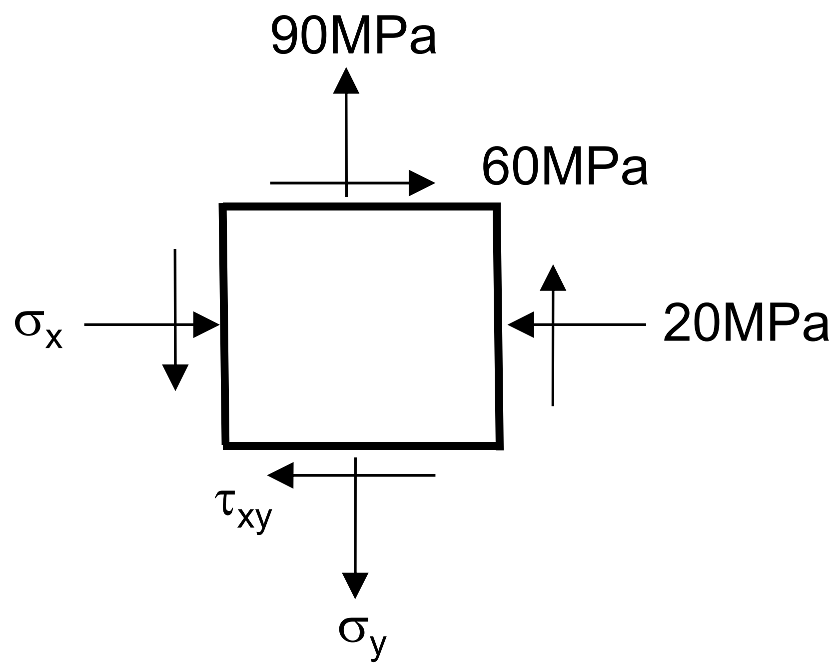
$$\tau_{x'y'} = -\frac{(-20-90)}{2}\sin(-47.49°) + 60\cos(-47.49°)$$

$$\tau_{x'y'} = 0$$

No shear stress acts on this element.

Example 2

The state of plane stress at a point on a body is represented on the element shown in the Figure. Represent this stress state in terms of the maximum in-plane shear stress and associated average normal stress.



Example 2 (cont.)

Maximum In-Plane Shear Stresses

$$\tau_{\max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\tau_{\max} = \pm\sqrt{\left(\frac{-20-90}{2}\right)^2 + 60^2}$$

$$\tau_{\max} = \pm 81.4 MPa$$

Example 2 (cont.)

Orientation of Element

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

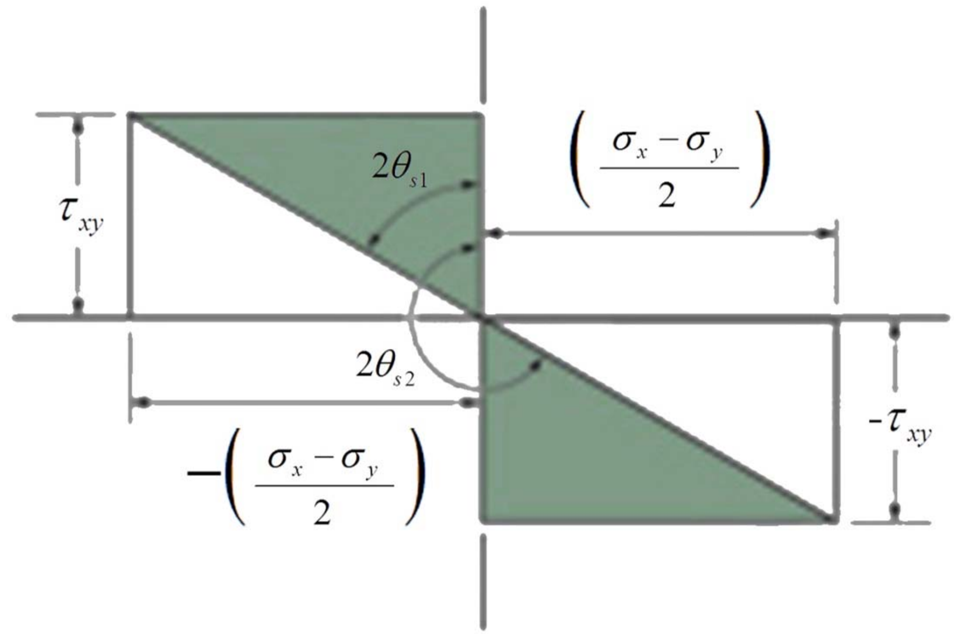
$$\tan 2\theta_s = -\frac{(-20-90)}{2\times 60}$$

$$2\theta_{s2} = 42.5°$$

$$\theta_{s2} = 21.3°$$

$$2\theta_{s1} = 180° + 42.5° = 222.5°$$

$$\theta_{s1} = 111.3°$$



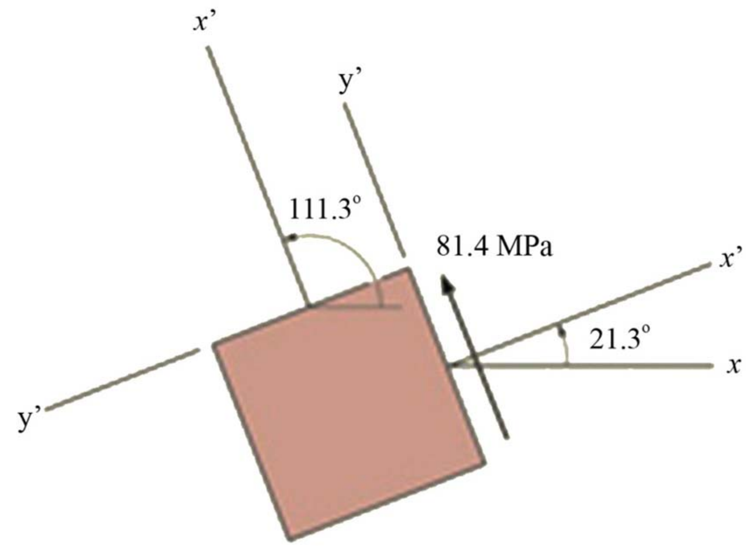
Worked Example

The proper direction of τmax on the element can be determined by applying the equation:

$$\tau_{x'y'} = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\tau_{x'y'} = -\frac{(-20 - 90)}{2}\sin 42.5° + 60\cos 42.5°$$

$$\tau_{x'y'} = 81.4MPa$$



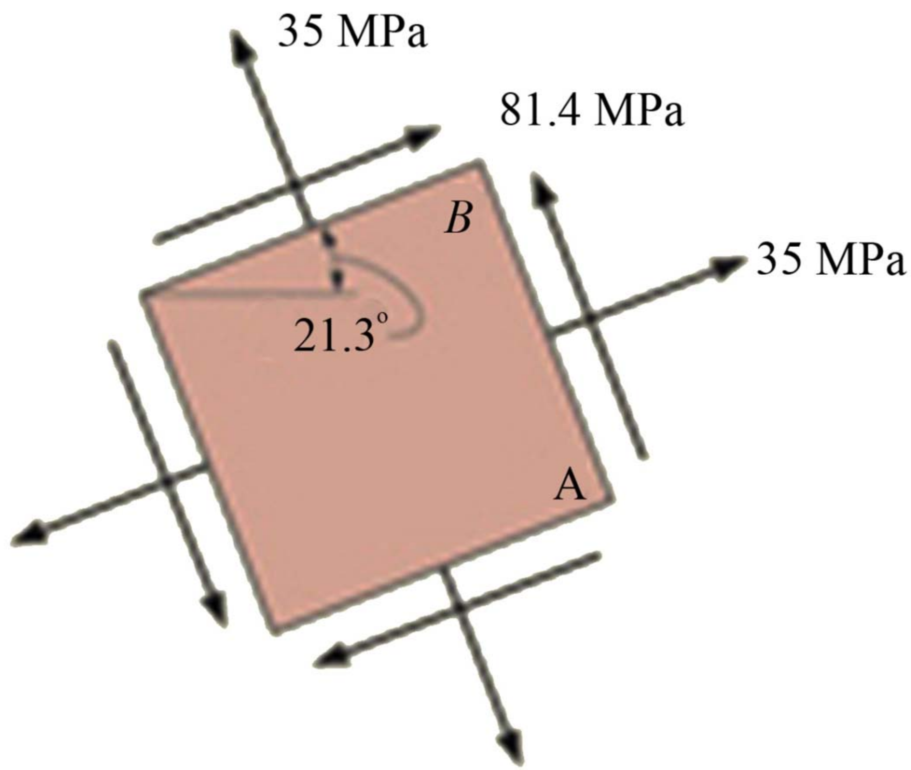
Example 2 (cont.)

Average Normal Stress. Besides the maximum shear stress, as calculated above, the element is also subjected to an average normal stress determined from the equation:

$$\sigma_{avg} = \frac{(\sigma_x + \sigma_y)}{2}$$

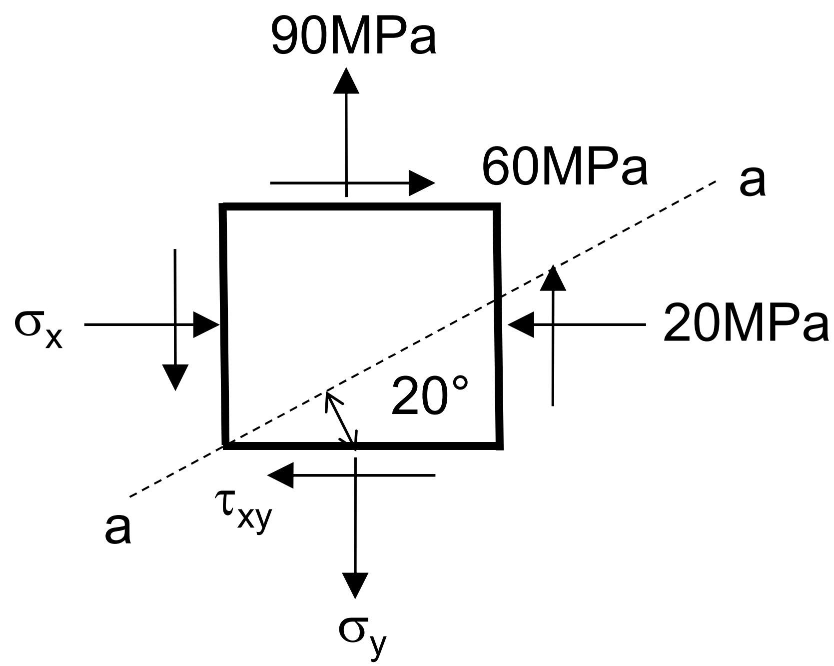
$$\sigma_{avg} = \frac{(-20 + 90)}{2}$$

$$\sigma_{avg} = 35MPa$$



Example 3

The state of plane stress at a point on a body is represented on the element shown in the Figure. Determine the stress components acting on the inclined plane a-a.



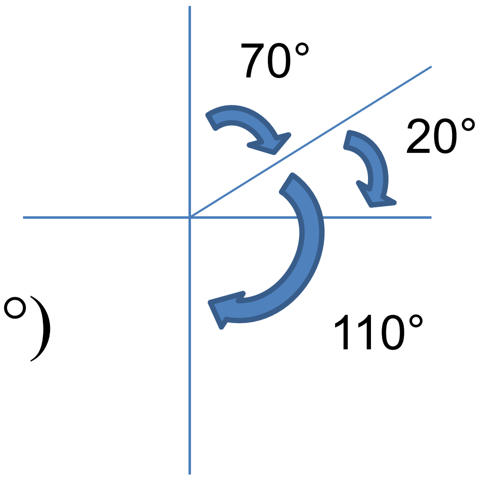
Example 3 (cont.)

Solution

Normal Stress:

$$\sigma_{x'} = \frac{(-20+90)}{2} + \frac{(-20-90)}{2}\cos(-70°) + 60\sin(-70°)$$

$$\sigma_{x'} = -2.57MPa$$ **(Compressive Stress)**

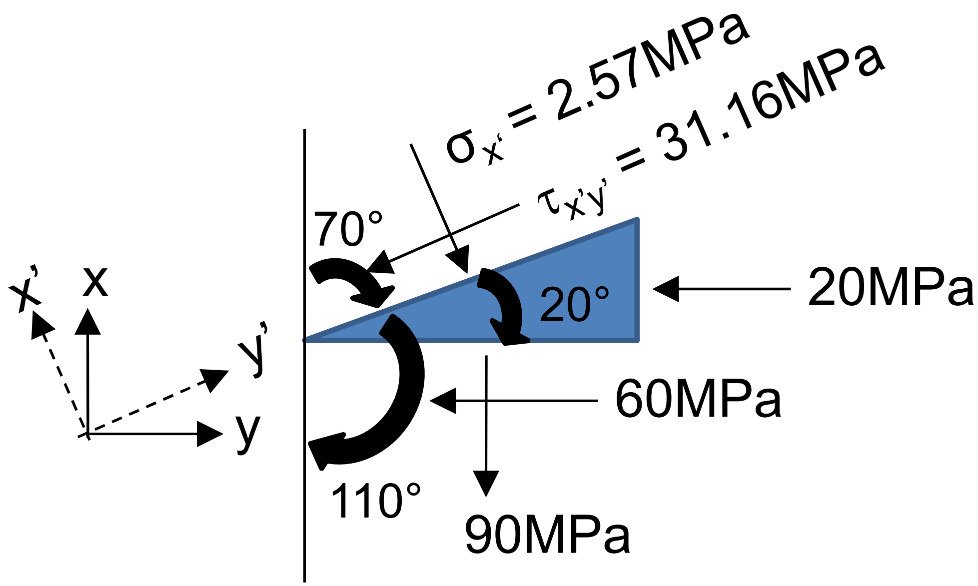


Shear Stress:

$$\tau_{x'y'} = -\frac{(-20-90)}{2}\sin(-70°) + 60\cos(-70°)$$

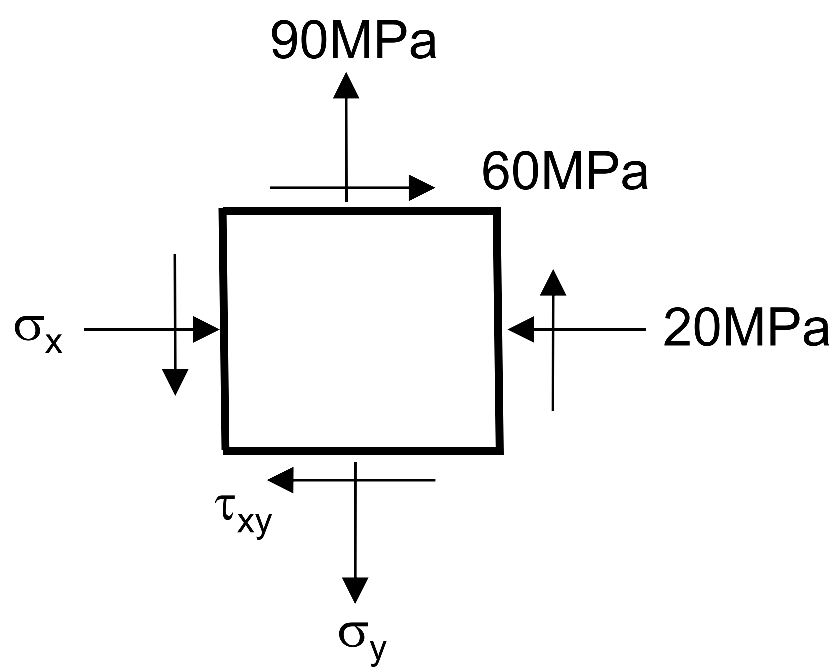
$$\tau_{x'y'} = -31.16MPa$$ **(Opposite Direction)**

Example 3 (cont.)



Example 4

The state of plane stress at a point on a body is represented on the element shown in the Figure. Determine the principal stresses acting at this point.



Example 4 (cont.)

Construction of the Circle.

σ_x = – 20MPa (Compression)

σ_y = 90MPa (Tension)

τ_{xy} = 60MPa (Clockwise)

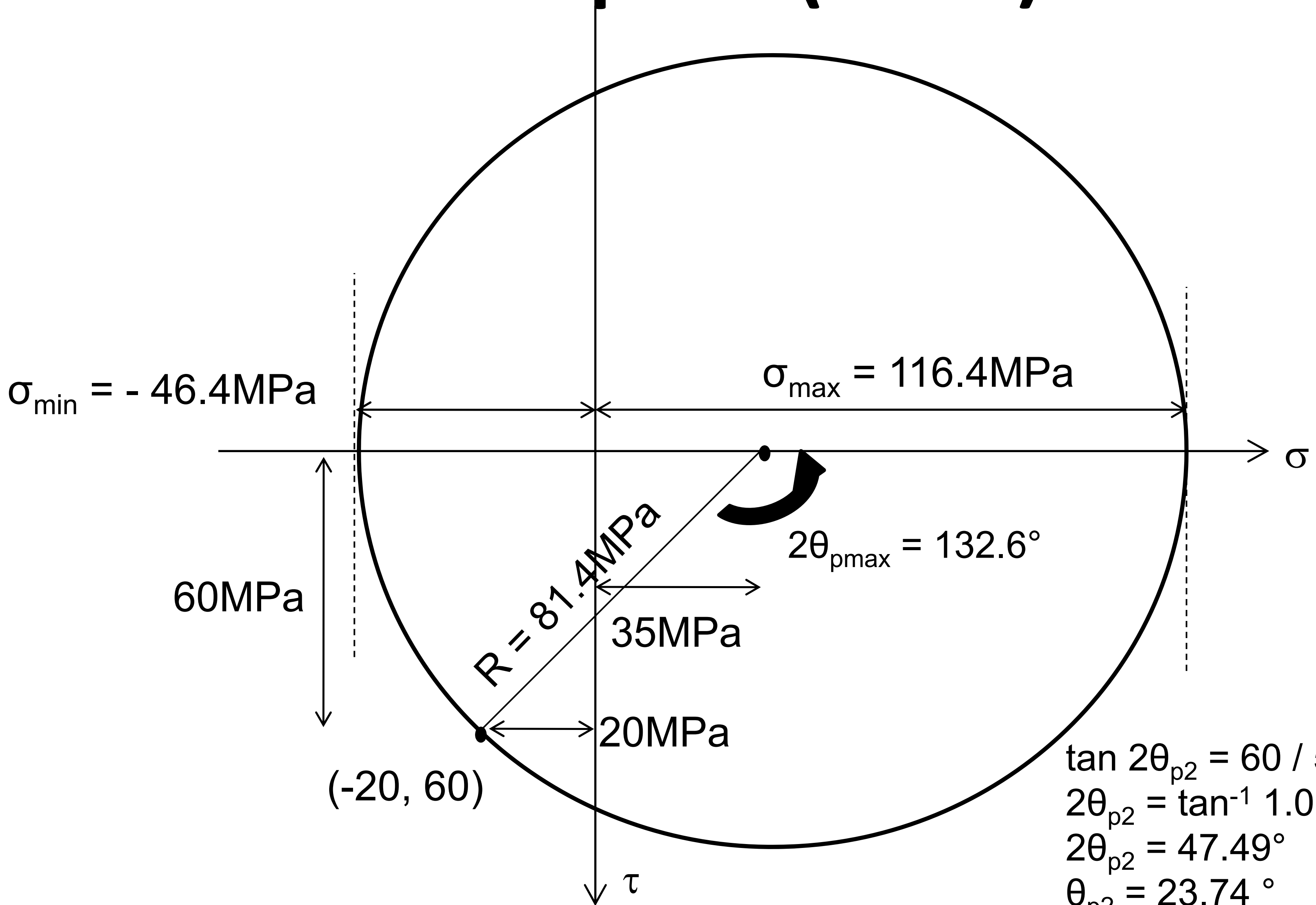
The Centre of Circle is at

$$\sigma_{avg} = \frac{(\sigma_x + \sigma_y)}{2} = \frac{(-20 + 90)}{2} = 35MPa$$

The Radius of the Circle is

$$R = \sqrt{\left[\frac{(\sigma_x - \sigma_y)}{2}\right]^2 + [\tau_{xy}]^2} = \sqrt{\left[\frac{(-20 - 90)}{2}\right]^2 + [60]^2} = 81.4MPa$$

Example 4 (cont.)



$\tan 2\theta_{p2} = 60 / 55$

$2\theta_{p2} = \tan^{-1} 1.09$

$2\theta_{p2} = 47.49°$

$\theta_{p2} = 23.74\ °$

Example 4 (cont.)

$\Psi = 80° - 33.69°$

$\Psi = 46.31\ °$

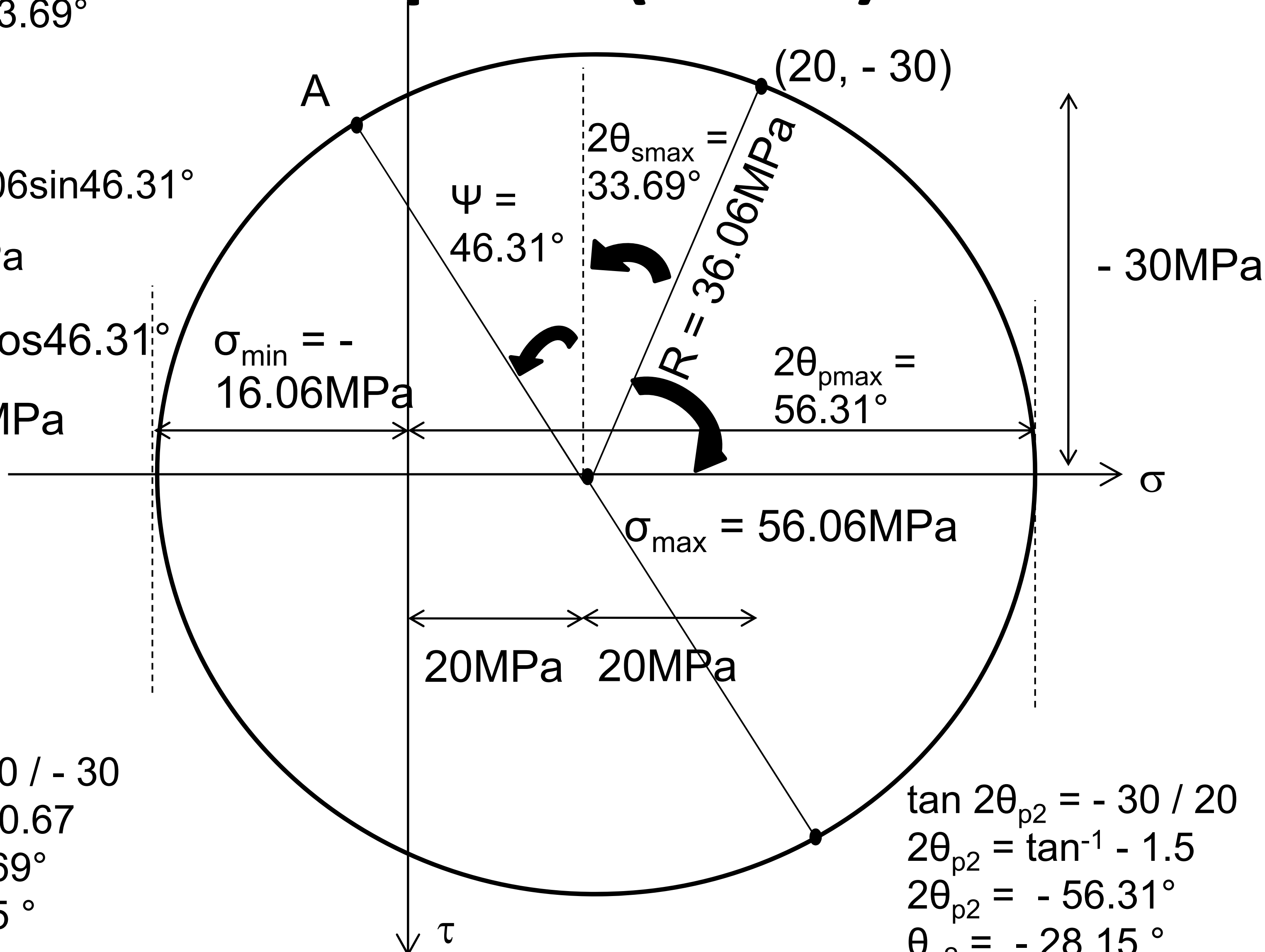
At Point A,

$\sigma_{x'} = 20 - 36.06 \sin 46.31°$

$\sigma_{x'} = -6.07\text{MPa}$

$\tau_{x'y'} = 36.06 \cos 46.31°$

$\tau_{x'y'} = 24.91\text{MPa}$



$\tan 2\theta_{s1} = 20 / -30$

$2\theta_{s1} = \tan^{-1} 0.67$

$2\theta_{s1} = -33.69°$

$\theta_{s1} = -16.85\ °$

$\tan 2\theta_{p2} = -30 / 20$

$2\theta_{p2} = \tan^{-1} -1.5$

$2\theta_{p2} = -56.31°$

$\theta_{p2} = -28.15\ °$

Example 4 (cont.)

$\Psi = 80° – 33.69°$

$\Psi = 46.31°$

At Point A,

$\sigma_{x'} = 20 – 36.06\sin 46.31°$

$\sigma_{x'} = -6.07\text{MPa}$

$\tau_{x'y'} = 36.06\cos 46.31°$

$\tau_{x'y'} = 24.91\text{MPa}$

$\tan 2\theta_{p2} = -30 / 20$

$2\theta_{p2} = \tan^{-1} -1.5$

$2\theta_{p2} = -56.31°$

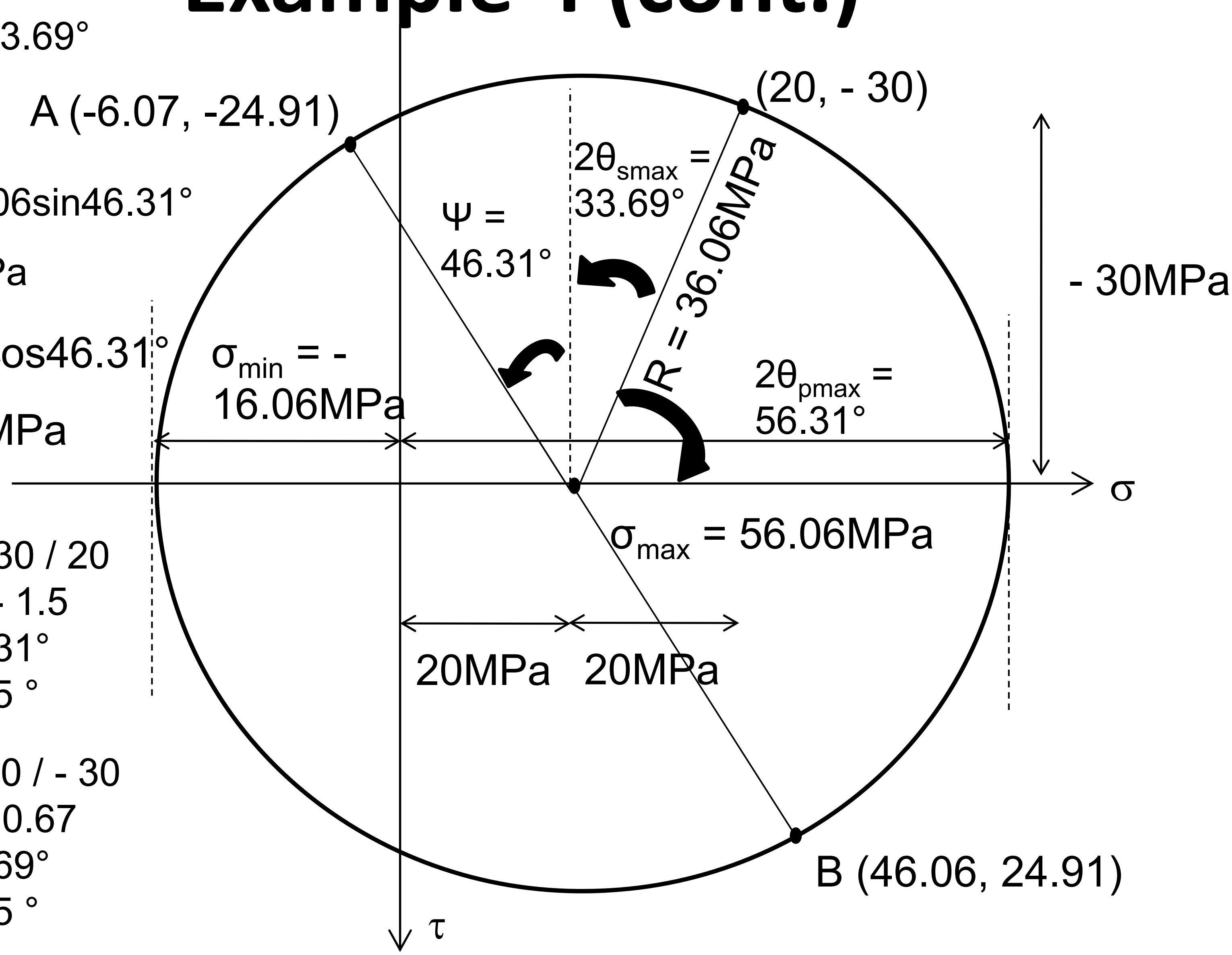
$\theta_{p2} = -28.15°$

$\tan 2\theta_{s1} = 20 / -30$

$2\theta_{s1} = \tan^{-1} 0.67$

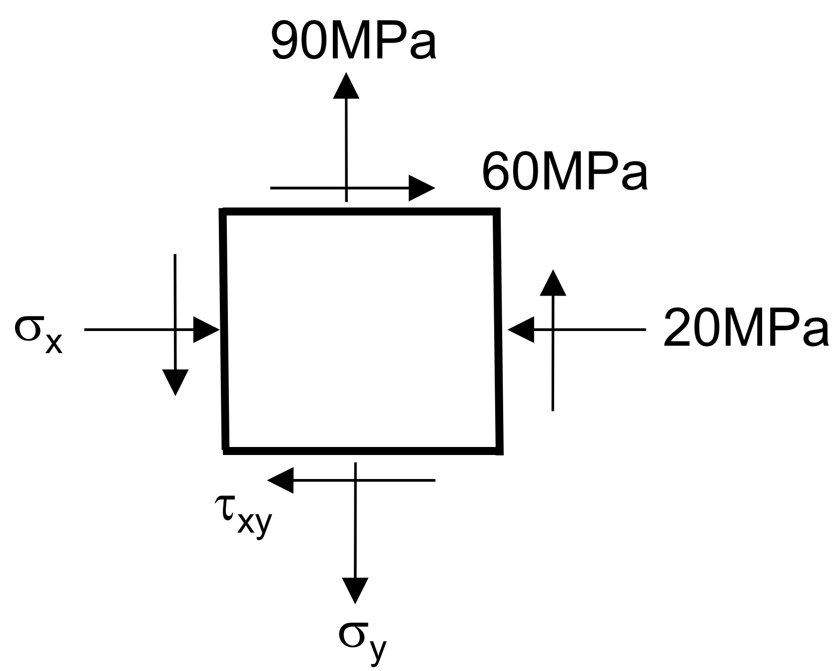
$2\theta_{s1} = -33.69°$

$\theta_{s1} = -16.85°$



Example 5

The state of plane stress at a point on a body is represented on the element shown in the Figure. Determine the maximum in-plane shear stresses and the orientation of the element upon which they act.



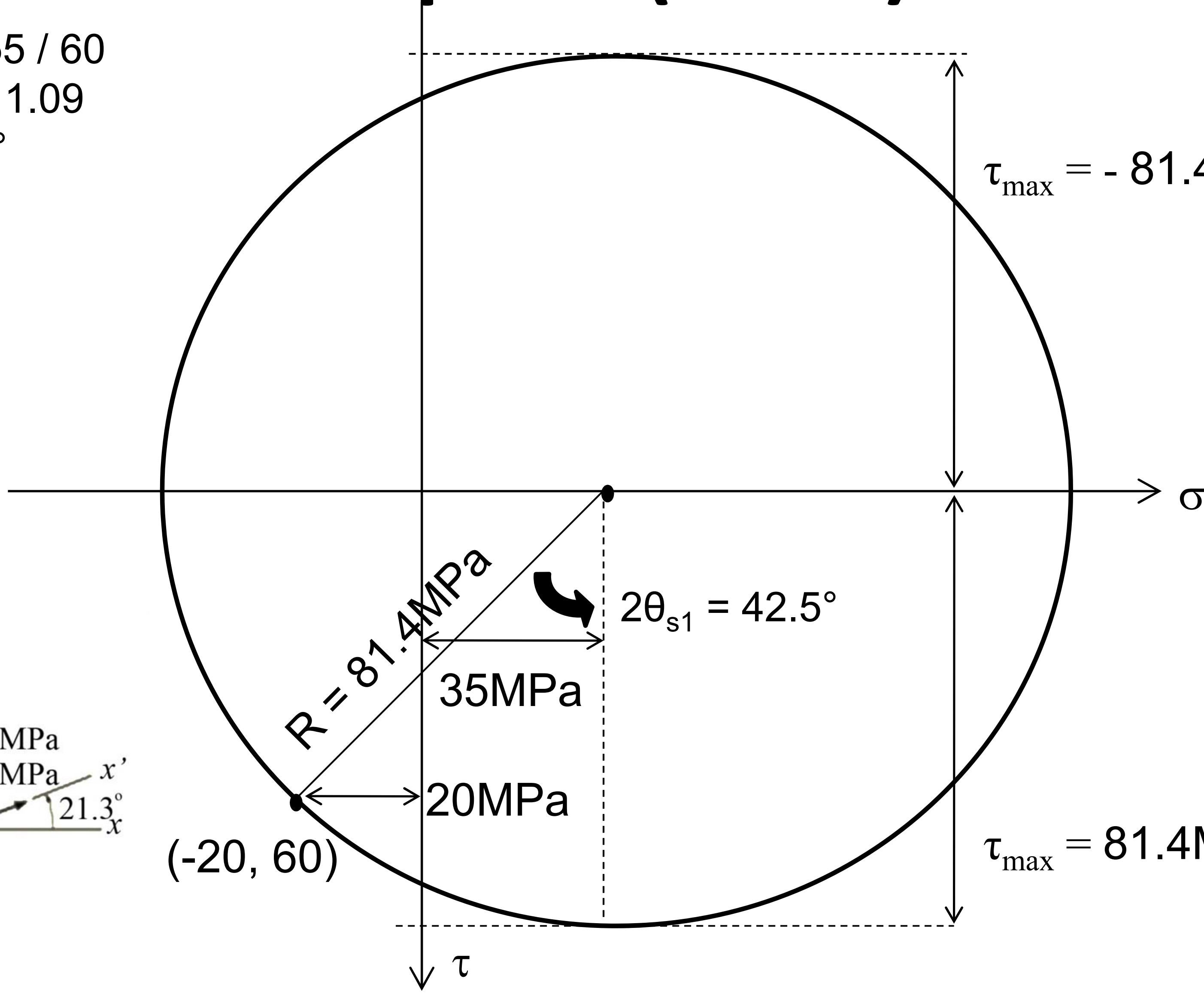
Example 5 (cont.)

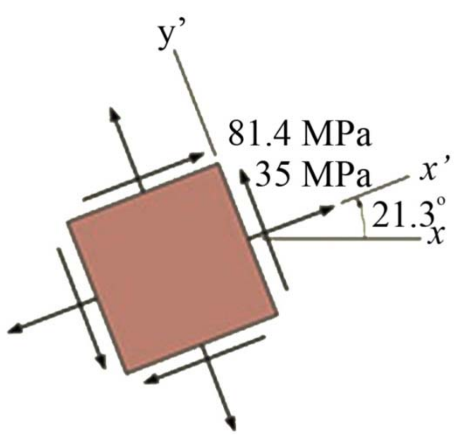
$\tan 2\theta_{s1} = 55 / 60$

$2\theta_{s1} = \tan^{-1} 1.09$

$2\theta_{s1} = 42.5°$

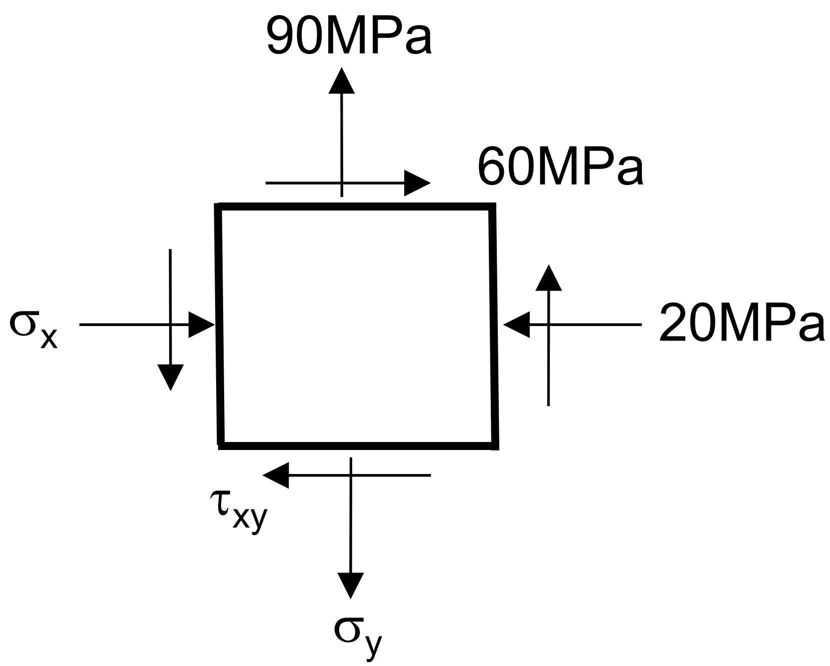
$\theta_{s1} = 21.3°$





Example 6

The state of plane stress at a point on a body is represented on the element shown in the Figure. Represent this state of stress on an element oriented 30° counterclockwise from the position shown.



Example 6 (cont.)

$\tan 2\theta = 55 / 60$
$2\theta = \tan^{-1} 1.09$
$2\theta = 42.5°$
$\theta = 21.3°$

$\Psi = 60° – 42.5°$
$\Psi = 17.5°$

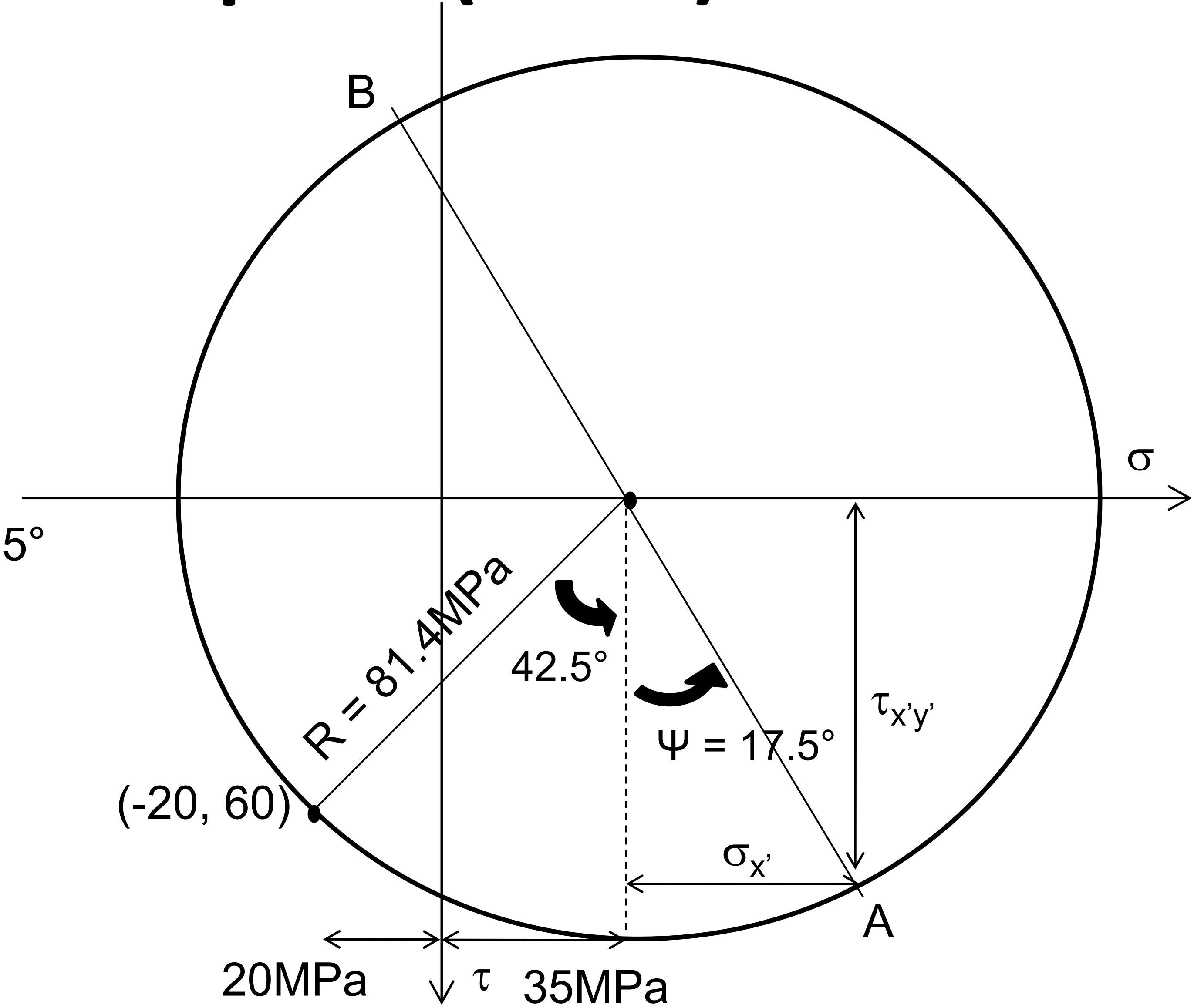
At Point A,

$\sigma_{x'} = 35 + 81.4\sin 17.5°$

$\sigma_{x'} = 59.48\text{MPa}$

$\tau_{x'y'} = 81.4\cos 17.5°$

$\tau_{x'y'} = 77.63\text{MPa}$



Example 6 (cont.)

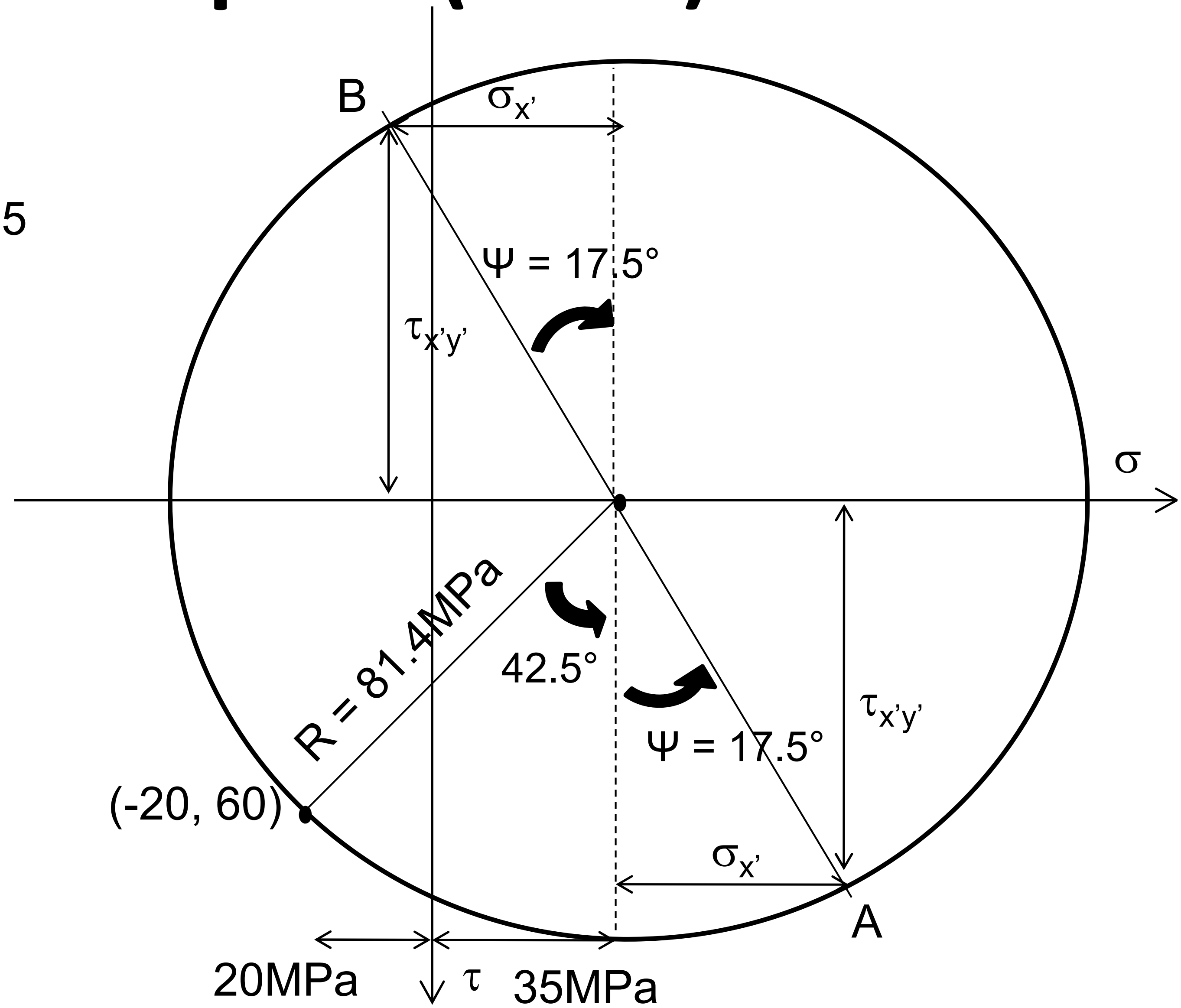
At Point B,

$\sigma_{x'} = 81.4\sin 17.5° - 35$

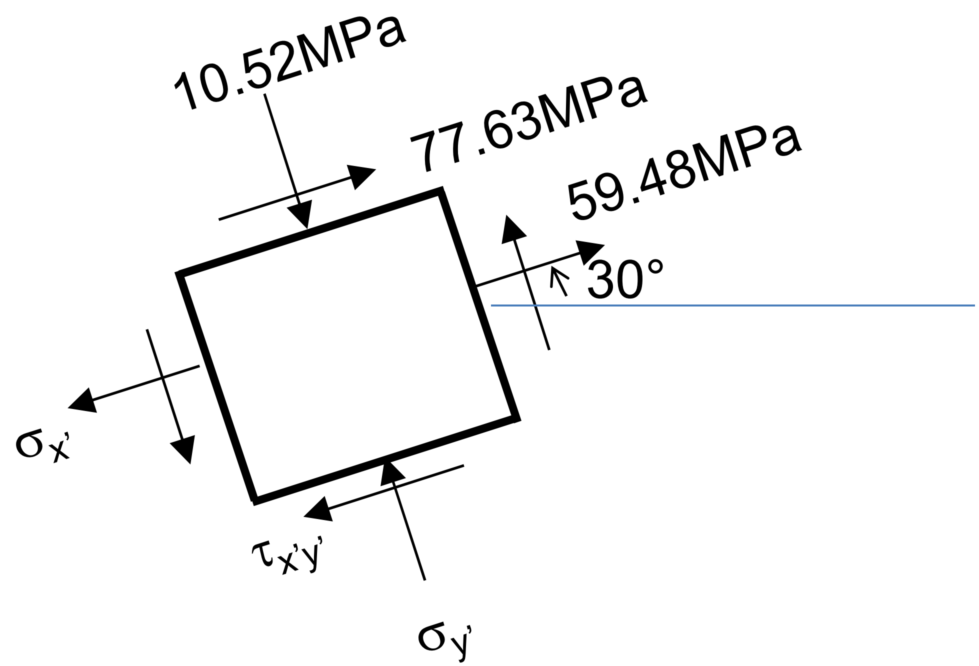
$\sigma_{x'} = -10.52\text{MPa}$

$\tau_{x'y'} = -81.4\cos 17.5°$

$\tau_{x'y'} = -77.63\text{MPa}$



Example 6 (cont.)



References

1. Hibbeler, R.C., Mechanics Of Materials, 8th Edition in SI units, Prentice Hall, 2011.
2. Gere dan Timoshenko, Mechanics of Materials, 3rd Edition, Chapman & Hall.
3. Yusof Ahmad, ‘Mekanik Bahan dan Struktur’ Penerbit UTM 2001