# SEB4233
# Biomedical Signal Processing

# Introduction to Wavelet

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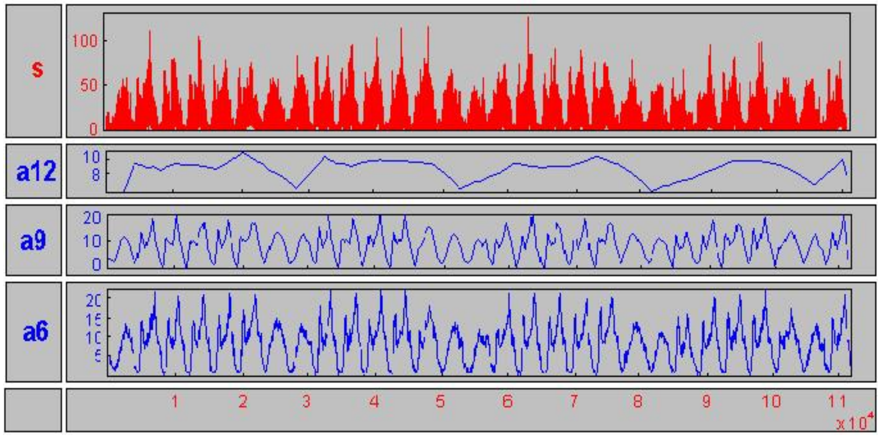
# Overview



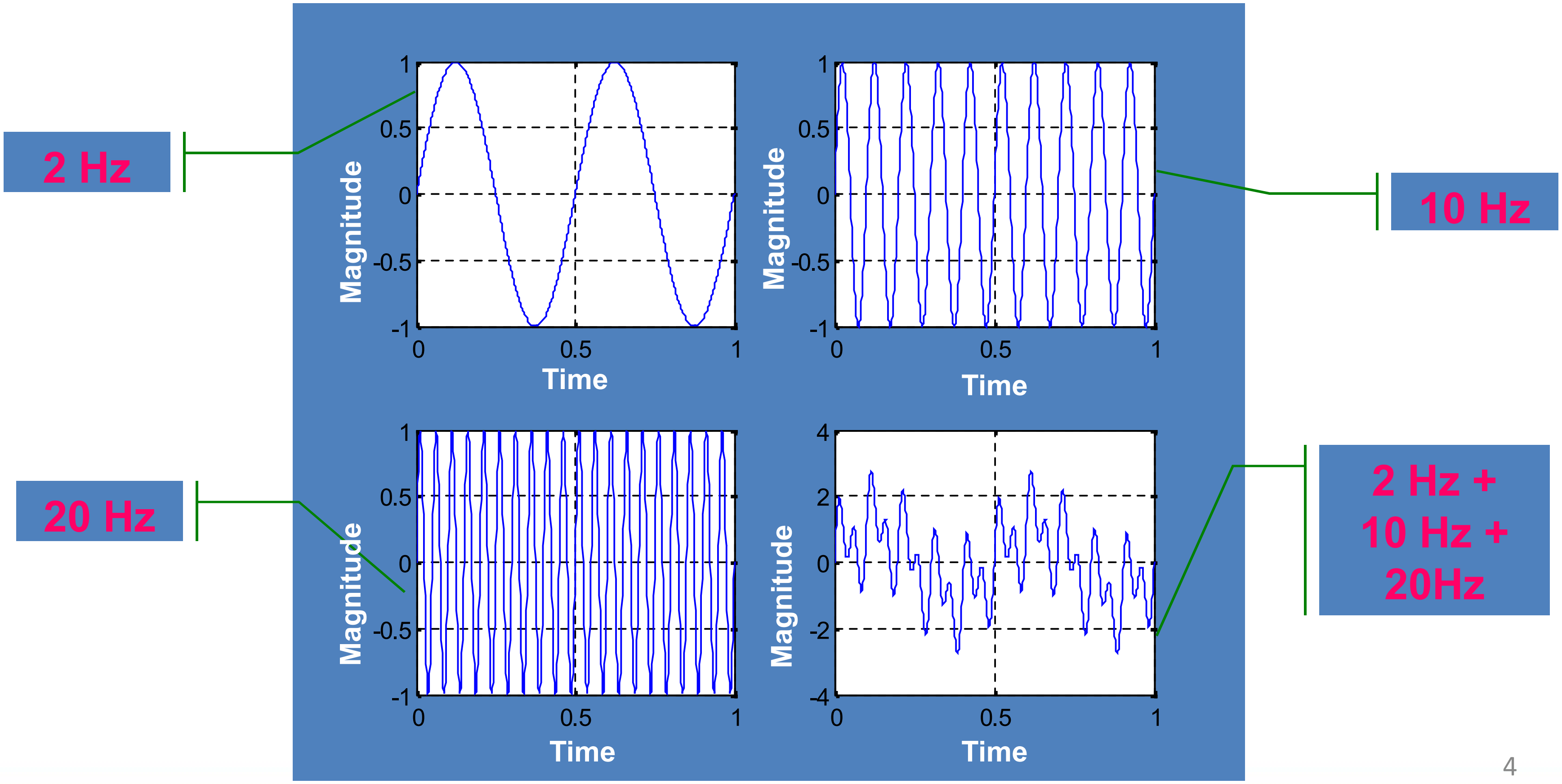

- Wavelet
  - A small wave
- Wavelet Transforms
  - Convert a signal into a series of wavelets
  - Provide a way for analyzing waveforms, bounded in both frequency and duration
  - Allow signals to be stored more efficiently than by Fourier transform
  - Be able to better approximate real-world signals
  - Well-suited for approximating data with sharp discontinuities
- "The Forest & the Trees"
  - Notice gross features with a large "window"
  - Notice small features with a small "window"

# Mathematical Transformation

- Why
  - To obtain a further information from the signal that is not readily available in the raw signal.
- Raw Signal
  - Normally the time-domain signal
- Processed Signal
  - A signal that has been "transformed" by any of the available mathematical transformations
- Fourier Transformation
  - The most popular transformation

# Time-Domain Signal

- The Independent Variable is Time
- The Dependent Variable is the Amplitude
- Most of the Information is Hidden in the Frequency Content



# Frequency Transforms

- Why Frequency Information is Needed
  - Be able to see any information that is not obvious in time-domain

- Types of Frequency Transformation
  - Fourier Transform, Hilbert Transform, Short-time Fourier Transform, Wigner Distributions, the Radon Transform, the Wavelet Transform …

# Frequency Analysis

- Frequency Spectrum
  - Be basically the frequency components (spectral components) of that signal
  - Show what frequencies exists in the signal

- Fourier Transform (FT)
  - One way to find the frequency content
  - Tells how much of each frequency exists in a signal

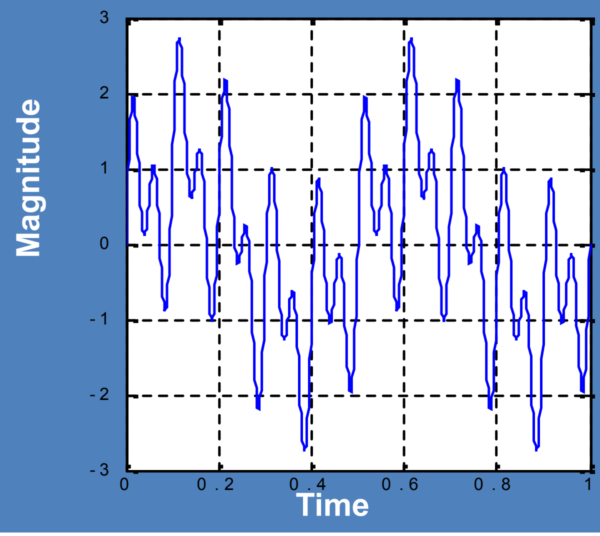
# Stationarity of Signal

- Stationary Signal
  - Signals with frequency content unchanged in time
  - All frequency components exist at all times
- Non-stationary Signal
  - Frequency changes in time

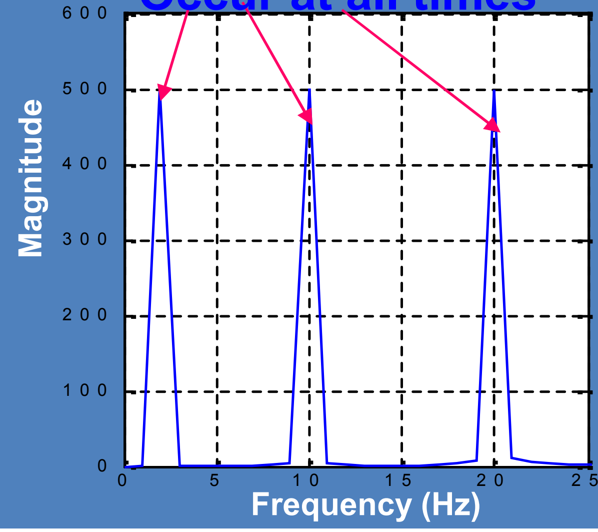
# Stationarity of Signal

**2 Hz + 10 Hz + 20Hz**

**Stationary**



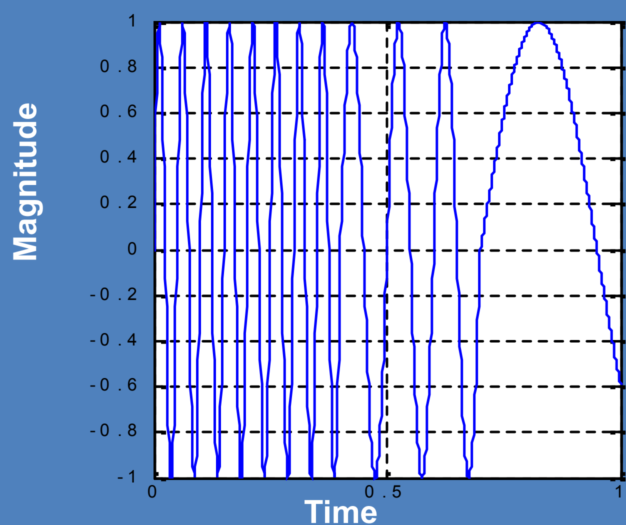

Occur at all times

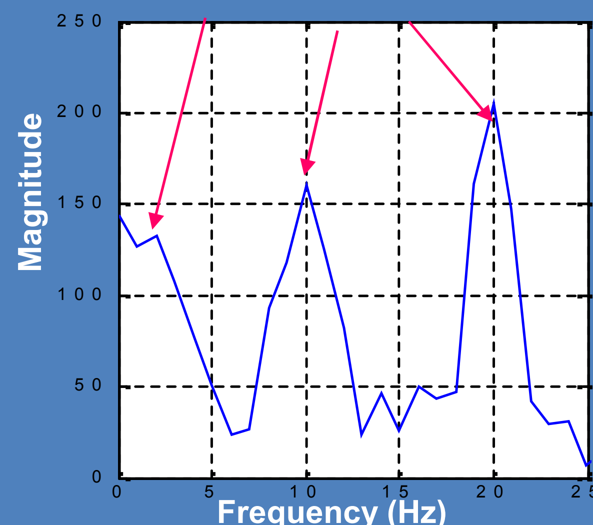



Do not appear at all times

**0.0-0.4: 2 Hz +**
**0.4-0.7: 10 Hz +**
**0.7-1.0: 20Hz**

**Non-Stationary**

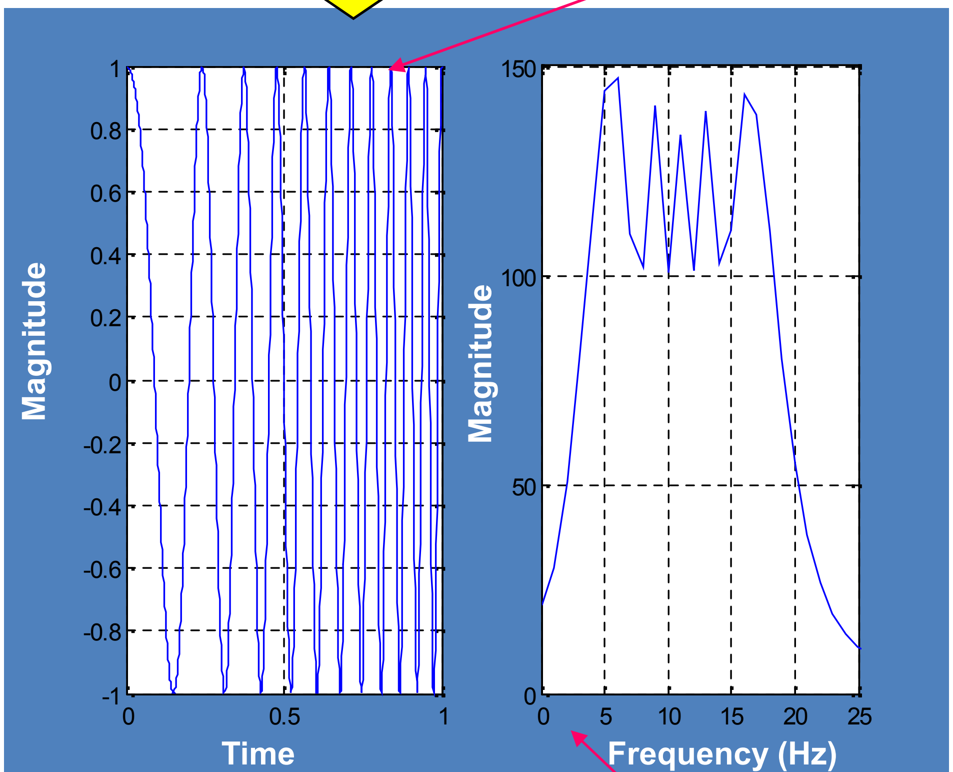


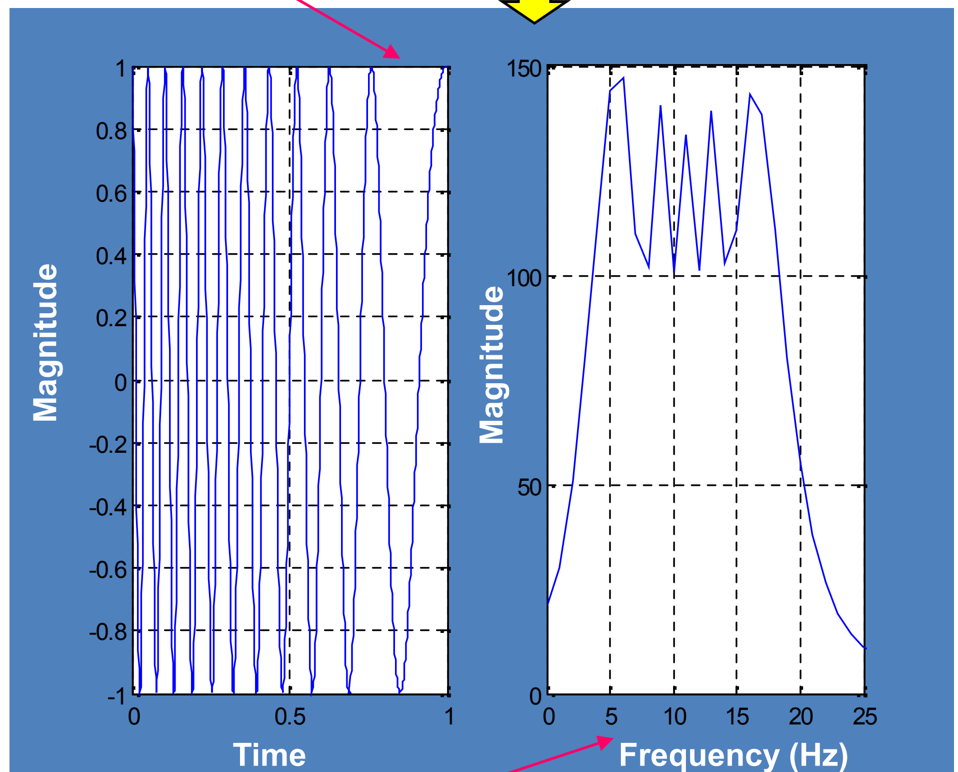



# Non-stationary Signals

- Frequency: 2 Hz to 20 Hz
- Frequency: 20 Hz to 2 Hz

**Different in Time Domain**




**Same in Frequency Domain**

**At what time the frequency components occur? FT can not tell!**

- FT Only Gives what Frequency Components Exist in the Signal
- The Time and Frequency Information can not be Seen at the Same Time
- Time-frequency Representation of the Signal is Needed

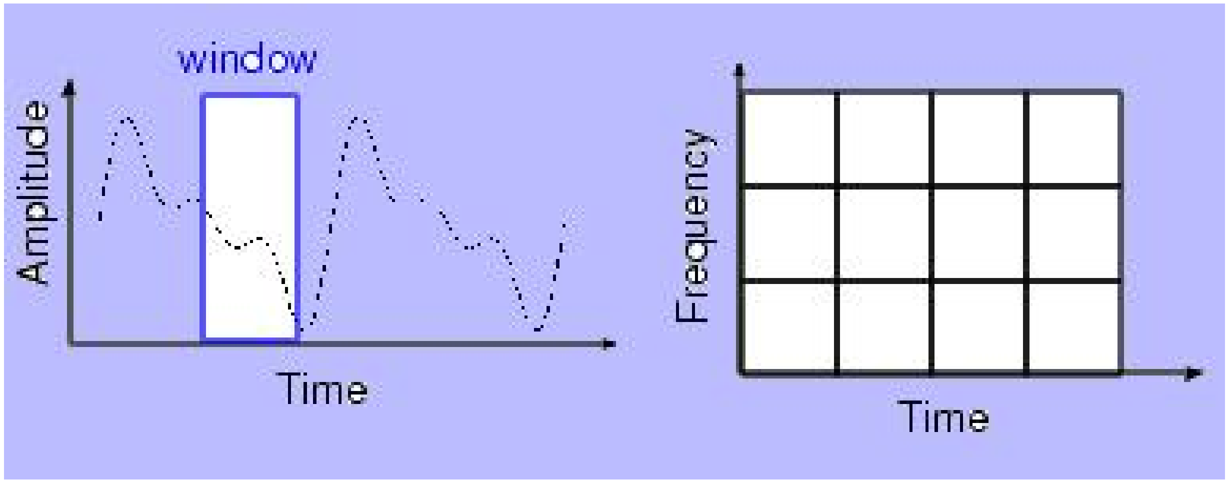
**Most of Transportation Signals are Non-stationary.**

**(We need to know whether and also when an incident was happened.)**

**ONE EARLIER SOLUTION: SHORT-TIME FOURIER TRANSFORM (STFT)**

# Short Time Fourier Transform (STFT)

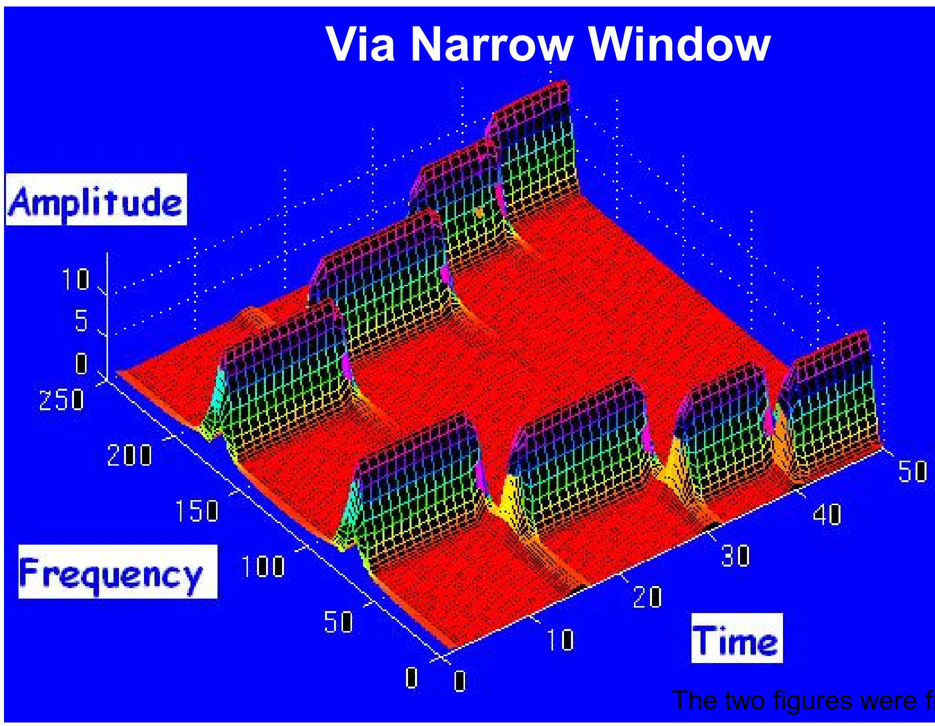
- Dennis Gabor (1946) Used STFT
  - To analyze only a small section of the signal at a time -- a technique called *Windowing the Signal*.
- The Segment of Signal is Assumed *Stationary*
- A 3D transform

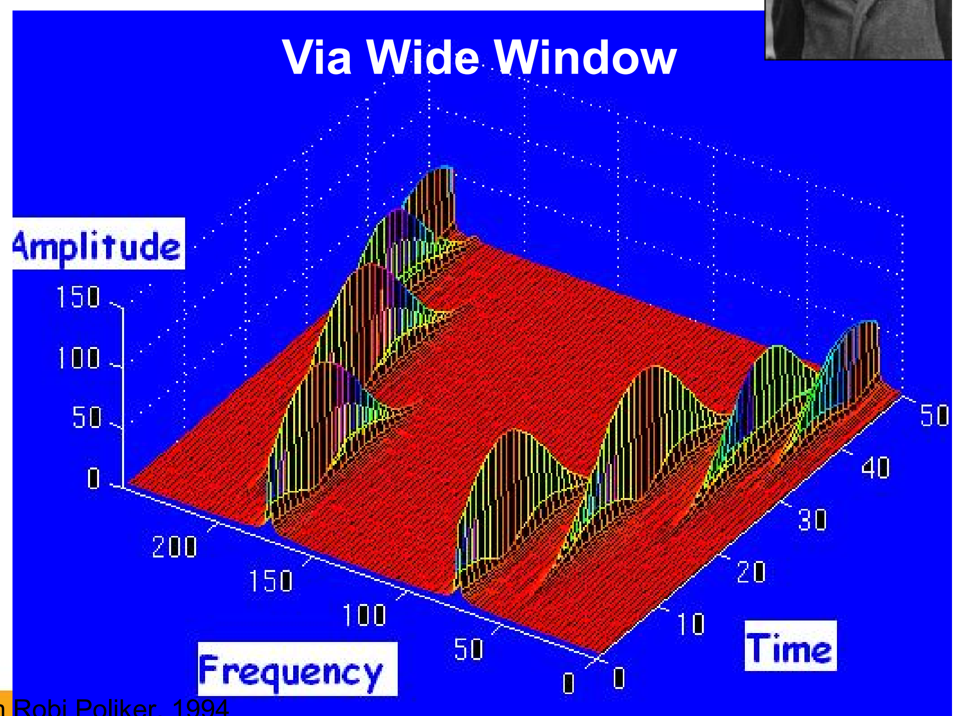



A function of time and frequency

# Drawbacks of STFT

- Unchanged Window
- Dilemma of Resolution
  - Narrow window -> poor frequency resolution
  - Wide window -> poor time resolution
- Heisenberg Uncertainty Principle
  - Cannot know what frequency exists at what time intervals




The two figures were from Robi Poliker, 1994

# Multiresolution Analysis (MRA)

- Wavelet Transform
  - An alternative approach to the short time Fourier transform to overcome the resolution problem
  - Similar to STFT: signal is multiplied with a function
- Multiresolution Analysis
  - Analyze the signal at different frequencies with different resolutions
  - Good time resolution and poor frequency resolution at high frequencies
  - Good frequency resolution and poor time resolution at low frequencies
  - More suitable for short duration of higher frequency; and longer duration of lower frequency components

# Advantages of WT over STFT

- Width of the Window is Changed as the Transform is Computed for Every Spectral Components
- Altered Resolutions are Placed

# Principles of Wavelet Transform

- Split Up the Signal into a Bunch of Signals
- Representing the Same Signal, but all Corresponding to Different Frequency Bands
- Only Providing What Frequency Bands Exists at What Time Intervals

# Definition of Continuous Wavelet Transform

Translation

(The location of the window)

Scale

Mother Wavelet

- Wavelet
  - Small wave
  - Means the window function is of finite length
- Mother Wavelet
  - A prototype for generating the other window functions
  - All the used windows are its dilated or compressed and shifted versions

# Scale

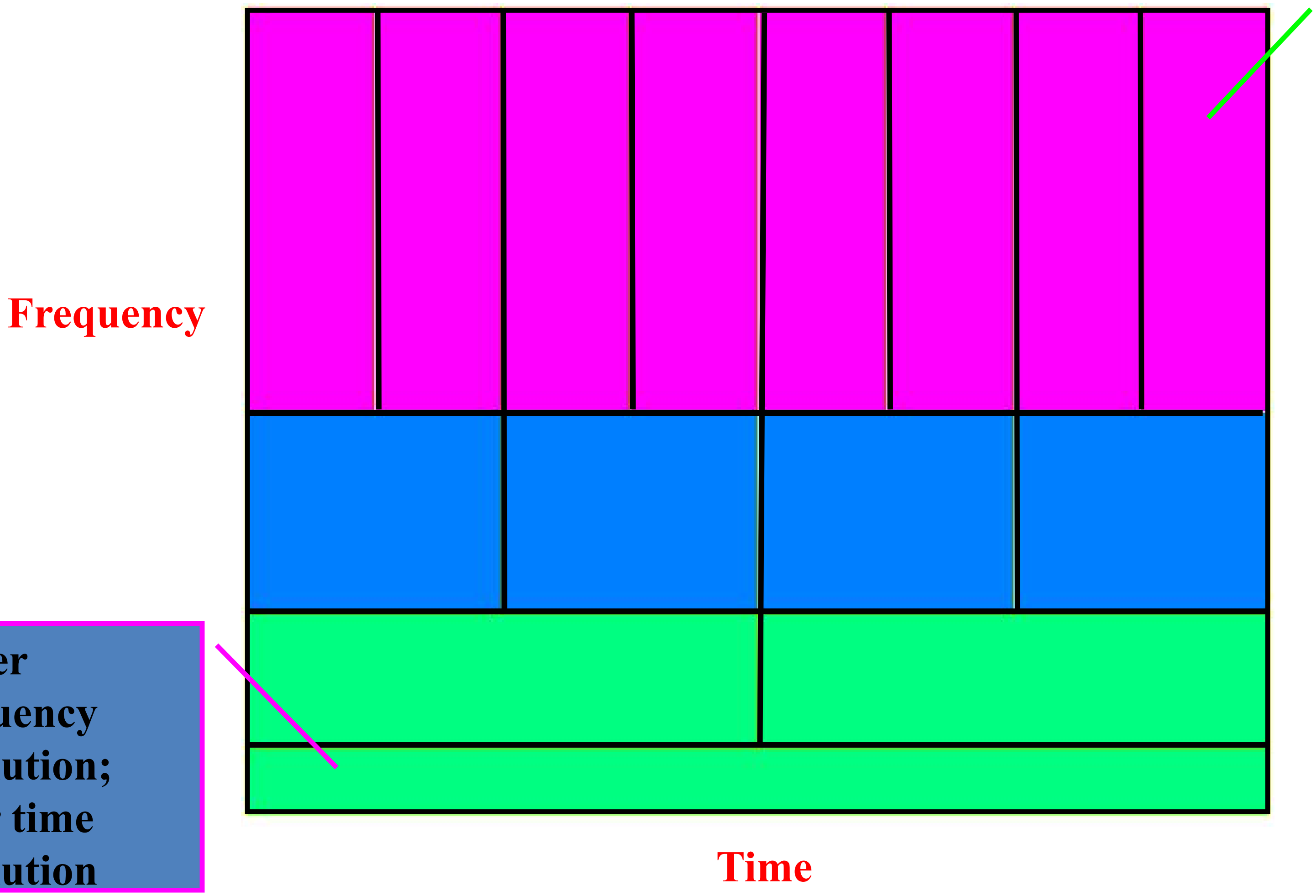
- Scale
  - S>1: dilate the signal
  - S<1: compress the signal
- Low Frequency -> High Scale -> Non-detailed Global View of Signal -> Span Entire Signal
- High Frequency -> Low Scale -> Detailed View Last in Short Time
- Only Limited Interval of Scales is Necessary

# Computation of CWT

$$CWT_x^{\psi}(\tau, s) = \Psi_x^{\psi}(\tau, s) = \frac{1}{\sqrt{|s|}} \int x(t) \bullet \psi^*\left(\frac{t-\tau}{s}\right) dt$$

- Step 1: The wavelet is placed at the beginning of the signal, and set $s$=1 (the most compressed wavelet);
- Step 2: The wavelet function at scale "1" is multiplied by the signal, and integrated over all times; then multiplied by τ;
- Step 3: Shift the wavelet to $t = 1/\sqrt{s}$ , and get the transform value at $t$=τ and $s$=1;
- Step 4: Repeat the procedure until the wavelet reaches the end of the signal;
- Step 5: Scale s is increased by a sufficiently small value, the above procedure is repeated for all $s$;
- Step 6: Each computation for a given $s$ fills the single row of the time-scale plane;
- Step 7: CWT is obtained if all $s$ are calculated.

# Resolution of Time and Frequency

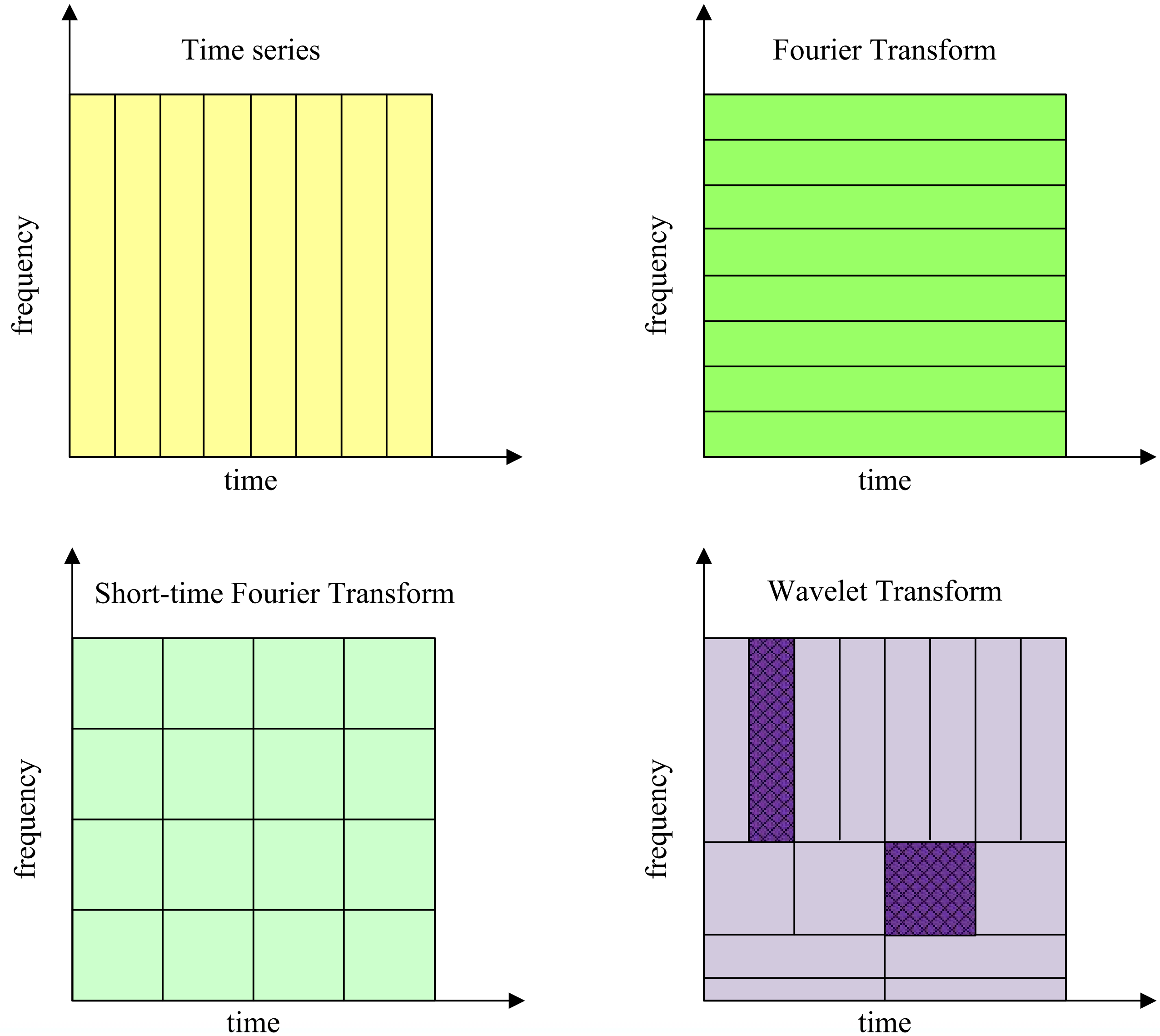



**Better time resolution; Poor frequency resolution**

**Better frequency resolution; Poor time resolution**

- Each box represents a equal portion
- Resolution in STFT is selected once for entire analysis

# Comparison of Transformations



# Mathematical Explanation

$$\psi^*_{\tau,s}(t)$$

CWT can be regarded as the inner product of the signal with a basis function

# Discretization of CWT

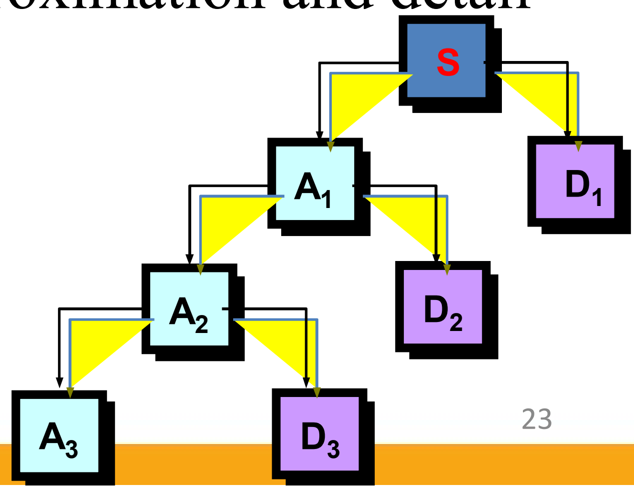
- It is Necessary to Sample the Time-Frequency (scale) Plane.
- At High Scale $s$ (Lower Frequency $f$), the Sampling Rate $N$ can be Decreased.
- The Scale Parameter $s$ is Normally Discretized on a Logarithmic Grid.
- The most Common Value is 2.

$$N_2 = s_1 / s_2 \cdot N_1 = f_1 / f_2 \cdot N_1$$

| S | 2 | 4 | 8 | … |
|---|---|---|---|---|
| N | 32 | 16 | 8 | … |

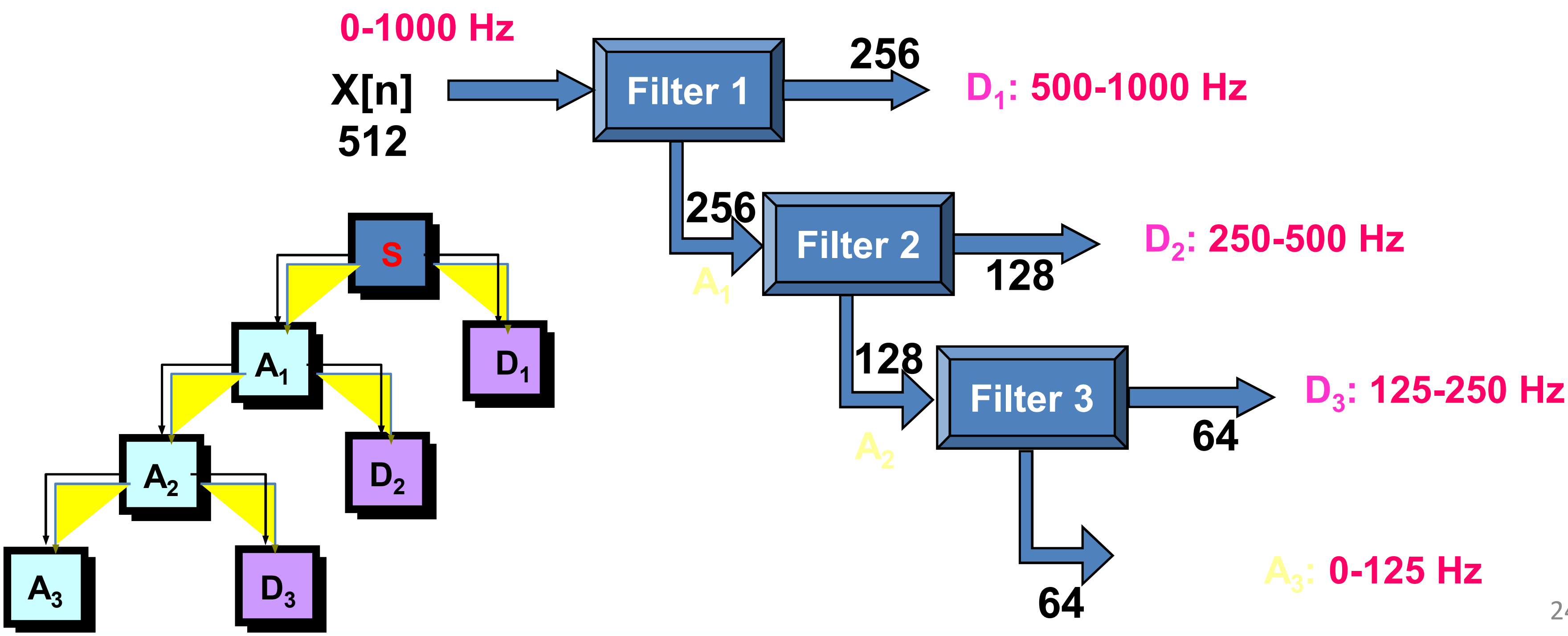
# Effective & Fast DWT

- The Discretized CWT is not a True Discrete Transform
- Discrete Wavelet Transform (DWT)
  - Provides sufficient information both for analysis and synthesis
  - Reduce the computation time sufficiently
  - Easier to implement
  - Analyze the signal at different frequency bands with different resolutions
  - Decompose the signal into a coarse approximation and detail information

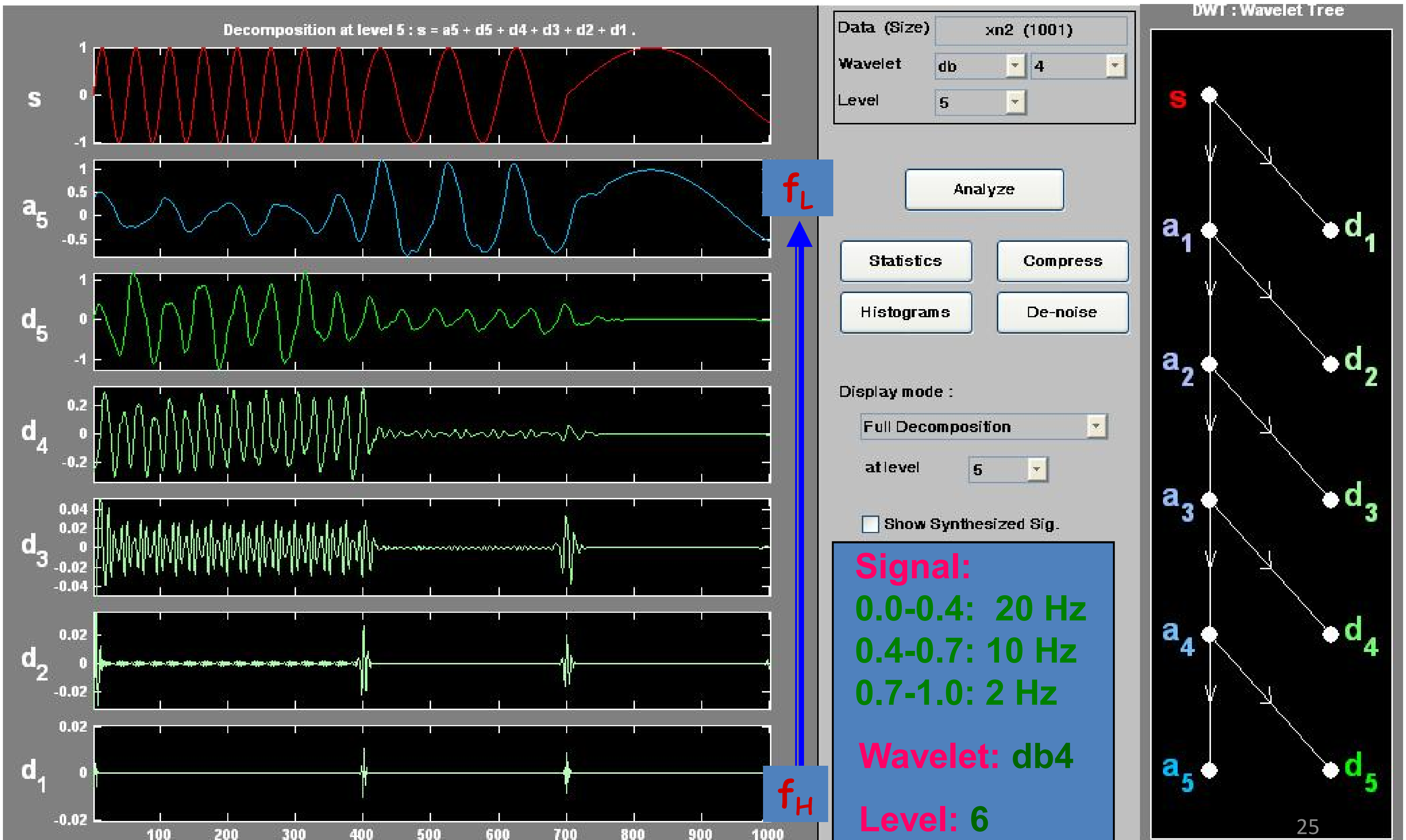


# SUBBABD Coding Algorithm

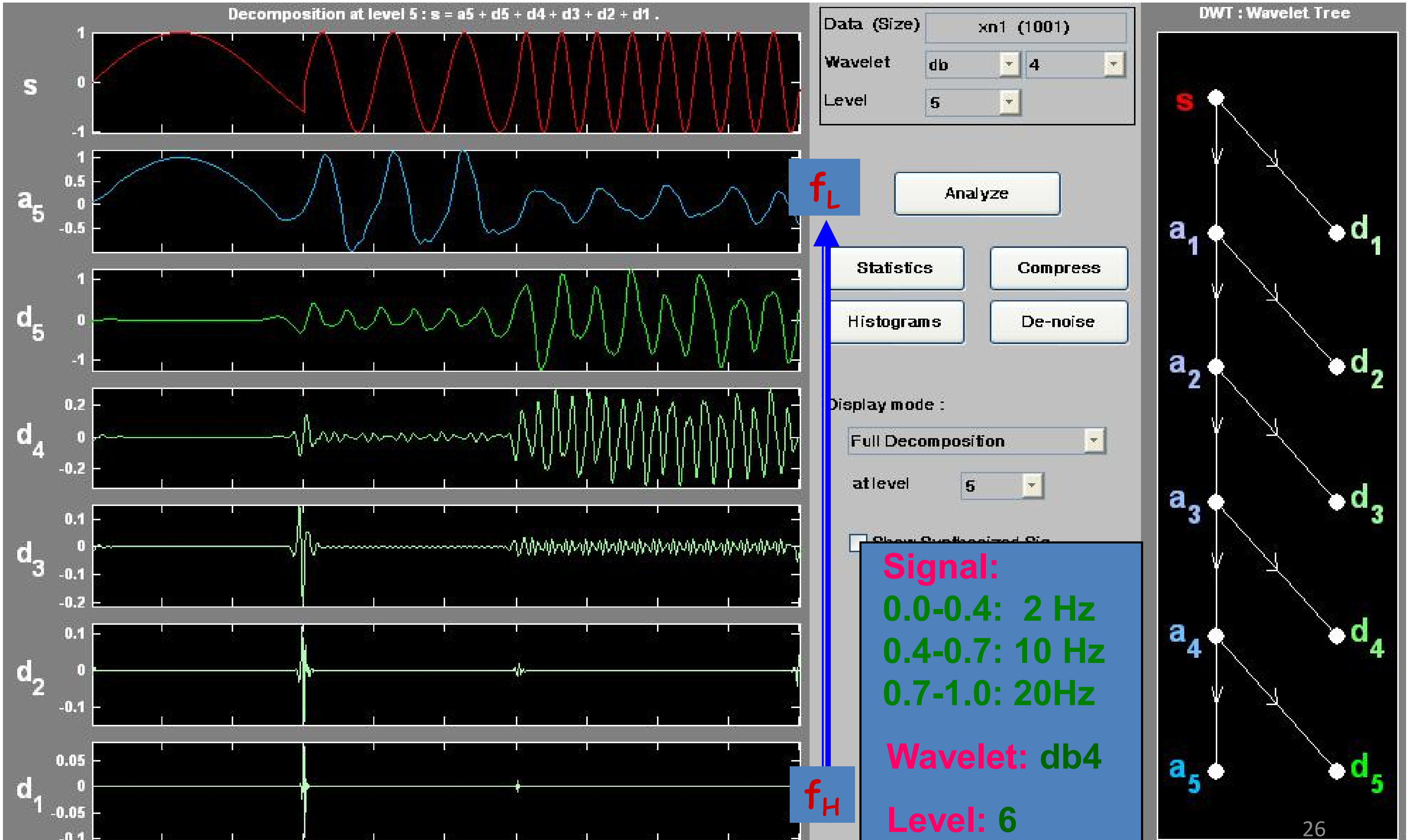
- Halves the Time Resolution
  - Only half number of samples resulted
- Doubles the Frequency Resolution
  - The spanned frequency band halved



# Decomposing Non-Stationary Signals (1)



# Decomposing Non-Stationary Signals (2)




Signal:
0.0-0.4: 2 Hz
0.4-0.7: 10 Hz
0.7-1.0: 20Hz

Wavelet: db4

Level: 6

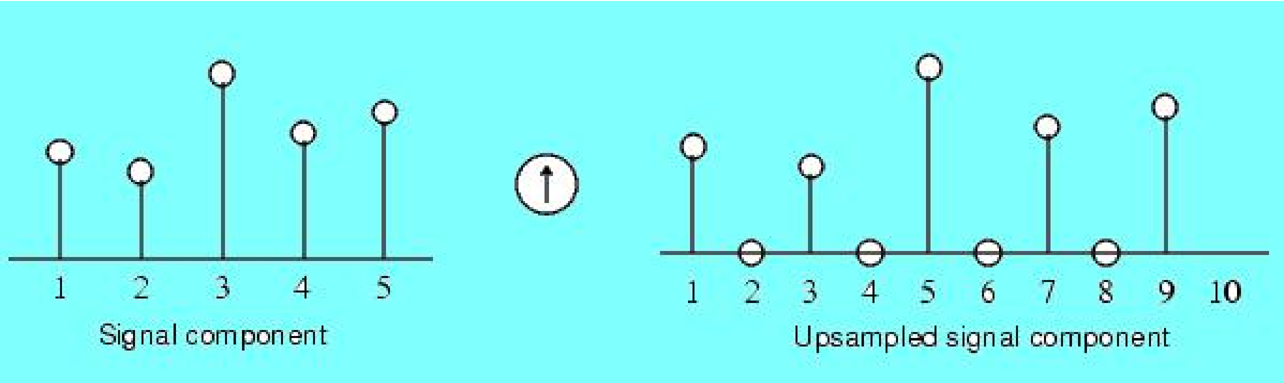
# Reconstruction (1)

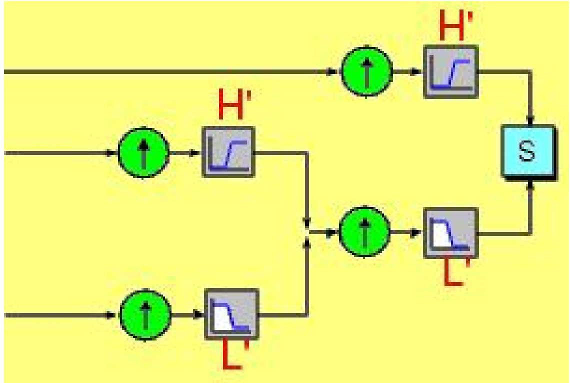
- What
  - How those components can be assembled back into the original signal without loss of information?
  - A Process After *decomposition* or *analysis*.
  - Also called *synthesis*

- How
  - Reconstruct the signal from the wavelet coefficients
  - Where wavelet analysis involves filtering and downsampling, the wavelet reconstruction process consists of upsampling and filtering

# Reconstruction (2)

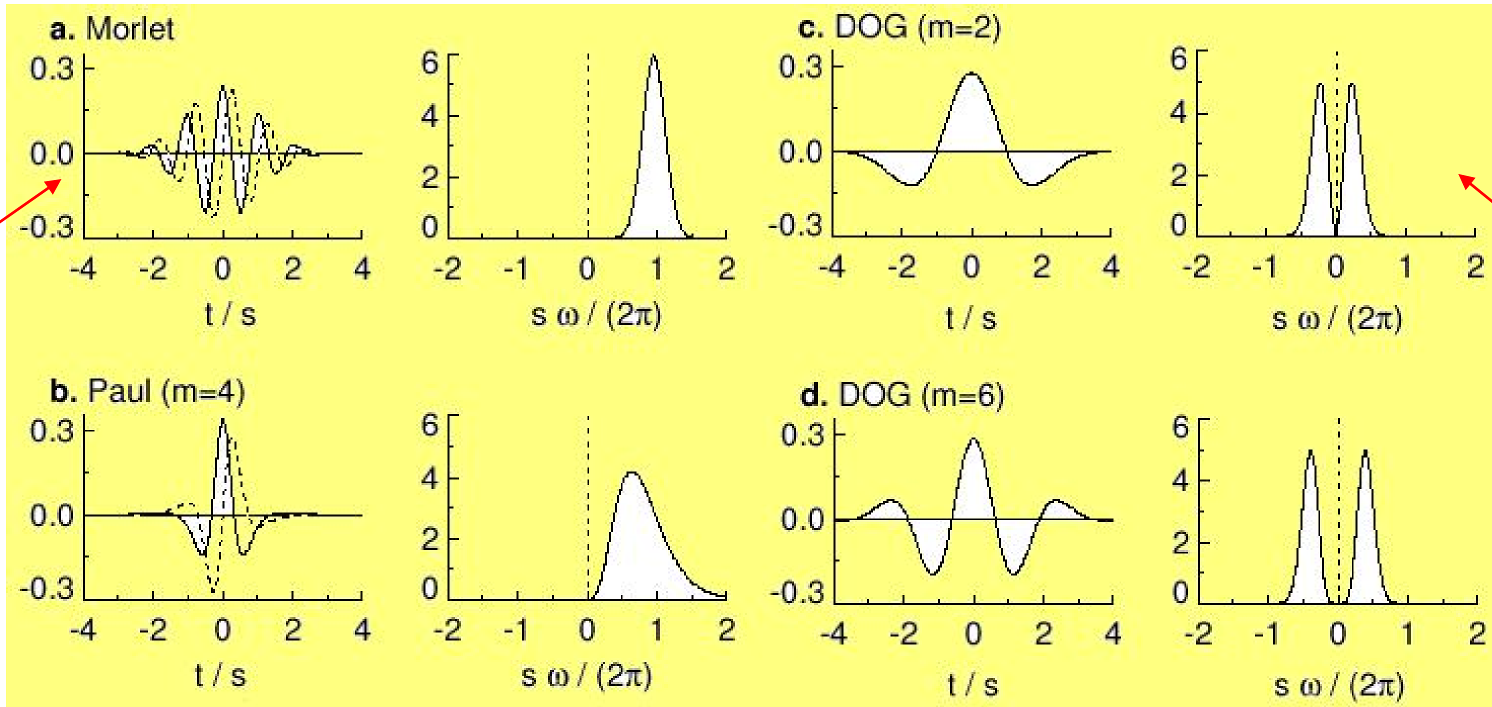
- Lengthening a signal component by inserting zeros between samples (upsampling)
- MATLAB Commands: idwt and waverec; idwt2 and waverec2.



# Wavelet Bases

Time domain




Frequency domain

Wavelet Basis Functions:

Morlet ($\omega_0$ = frequency): $\pi^{-1/4} e^{j\omega_0\eta} e^{-\eta^2/2}$

Paul ($m$ = order): DOG $\dfrac{2^m i^m m!}{\sqrt{\pi(2m)!}}(1-i\eta)^{-(m+1)}$

DOG ($m$ = devivative): $\dfrac{(-1)^{m+1}}{\sqrt{\Gamma\left(m+\frac{1}{2}\right)}}\dfrac{d^m}{d\eta^m}\left(e^{-\eta^2/2}\right)$

Derivative Of a Gaussian
M=2 is the **Marr** or **Mexican hat** wavelet

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# Wavelet Family Properties

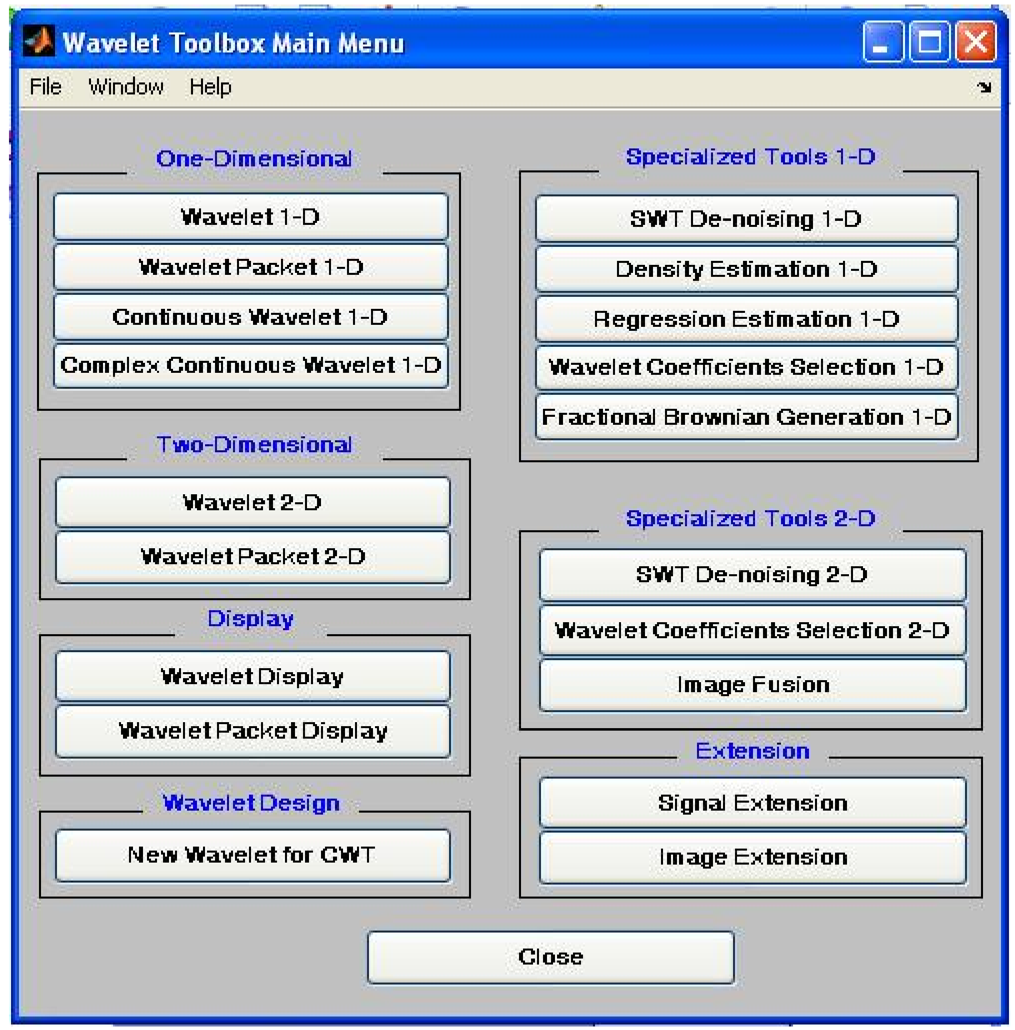
| Property | morl | mexh | meyr | haar | *db*N | symN | coifN | biorNr.Nd | rbioNr.Nd | gaus | dmey | cgau | cmor | fbsp | shan |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Crude** | ● | ● | | | | | | | | ● | | ● | ● | ● | ● |
| **Infinitely regular** | ● | ● | ● | | | | | | | ● | | ● | ● | ● | ● |
| **Arbitrary regularity** | | | | | ● | ● | ● | ● | ● | | | | | | |
| **Compactly supported orthogonal** | | | | ● | ● | ● | ● | | | | | | | | |
| **Compactly supported biothogonal** | | | | | | | | ● | ● | | | | | | |
| **Symmetry** | ● | ● | ● | ● | | | | ● | ● | ● | ● | ● | ● | ● | ● |
| **Asymmetry** | | | | | ● | | | | | | | | | | |
| **Near symmetry** | | | | | | ● | ● | | | | | | | | |
| **Arbitrary number of vanishing moments** | | | | | ● | ● | ● | ● | ● | | | | | | |
| **Vanishing moments for** $\phi$ | | | | | | | ● | | | | | | | | |
| **Existence of** $\phi$ | | | ● | ● | ● | ● | ● | ● | ● | | | | | | |
| **Orthogonal analysis** | | | ● | ● | ● | ● | ● | | | | | | | | |
| **Biorthogonal analysis** | | | ● | ● | ● | ● | ● | ● | ● | | | | | | |
| **Exact reconstruction** | ≈ | ● | ● | ● | ● | ● | ● | ● | ● | ● | ≈ | ● | ● | ● | ● |
| **FIR filters** | | | | ● | ● | ● | ● | ● | ● | | ● | | | | |
| **Continuous transform** | ● | ● | ● | ● | ● | ● | ● | ● | ● | ● | | | | | |
| **Discrete transform** | | | ● | ● | ● | ● | ● | ● | ● | | ● | | | | |
| **Fast algorithm** | | | | ● | ● | ● | ● | ● | ● | | ● | | | | |
| **Explicit expression** | ● | ● | | ● | | | | For splines | For splines | ● | | ● | ● | ● | ● |
| **Complex valued** | | | | | | | | | | | | ● | ● | ● | ● |
| **Complex continuous transform** | | | | | | | | | | | | ● | ● | ● | ● |
| **FIR-based approximation** | | | | | | | | | | | ● | | | | |

# Wavelet Software

- A Lot of Toolboxes and Software have been Developed
- One of the Most Popular Ones is the MATLAB Wavelet Toolbox
  http://www.mathworks.com/access/helpdesk/help/toolbox/wavelet/wavelet.html

# GUI Version in Matlab

- Graphical User Interfaces
- From the MATLAB prompt, type wavemenu, the Wavelet Toolbox Main Menu appears



# Other Software Sources

- WaveLib [http://www-sim.int-evry.fr/~bourges/WaveLib.html]
- EPIC [http://www.cis.upenn.edu/~eero/epic.html]
- Imager Wavelet Library [http://www.cs.ubc.ca/nest/imager/contributions/bobl/wvlt/download/]
- Mathematica wavelet programs [http://timna.Mines.EDU/wavelets/]
- Morletpackage [ftp://ftp.nosc.mil/pub/Shensa/]
- p-wavelets [ftp://pandemonium.physics.missouri.edu/pub/wavelets/]
- WaveLab [http://playfair.Stanford.EDU/~wavelab/]
- Rice Wavelet Tools [http://jazz.rice.edu/RWT/]
- Uvi_Wave Software [http://www.tsc.uvigo.es/~wavelets/uvi_wave.html]
- WAVBOX [ftp://simplicity.stanford.edu/pub/taswell/]
- Wavecompress [ftp://ftp.nosc.mil/pub/Shensa/]
- WaveThresh [http://www.stats.bris.ac.uk/pub/software/wavethresh/WaveThresh.html]
- WPLIB [ftp://pascal.math.yale.edu/pub/wavelets/software/wplib/]
- W-Transform Matlab Toolbox [ftp://info.mcs.anl.gov/pub/W-transform/]
- XWPL [ftp://pascal.math.yale.edu/pub/wavelets/software/xwpl/]
- …

# Wavelet Applications

- Typical Application Fields
  - Astronomy, acoustics, nuclear engineering, sub-band coding, signal and image processing, neurophysiology, music, magnetic resonance imaging, speech discrimination, optics, fractals, turbulence, earthquake-prediction, radar, human vision, and pure mathematics applications
- Sample Applications
  - Identifying pure frequencies
  - De-noising signals
  - Detecting discontinuities and breakdown points
  - Detecting self-similarity
  - Compressing images

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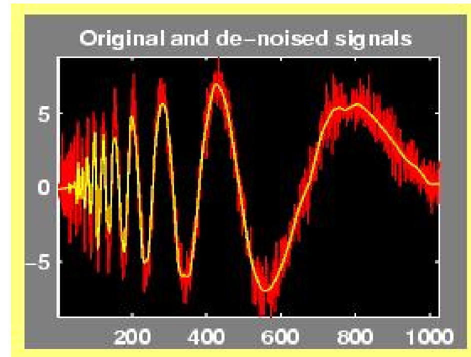
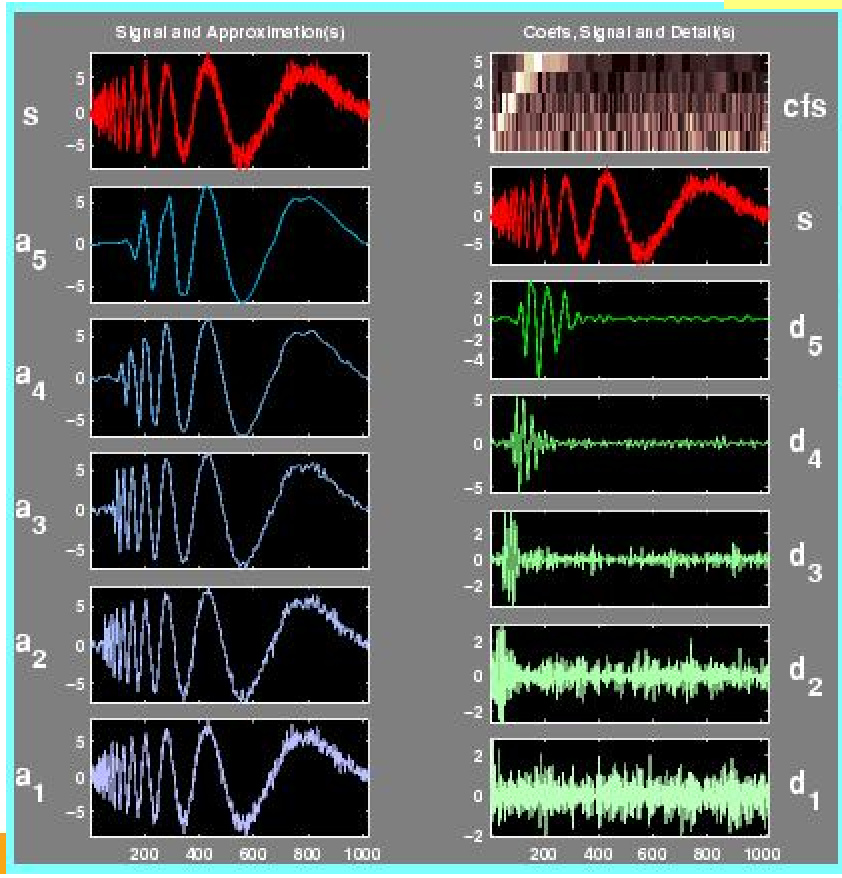
# De-Noising Signals

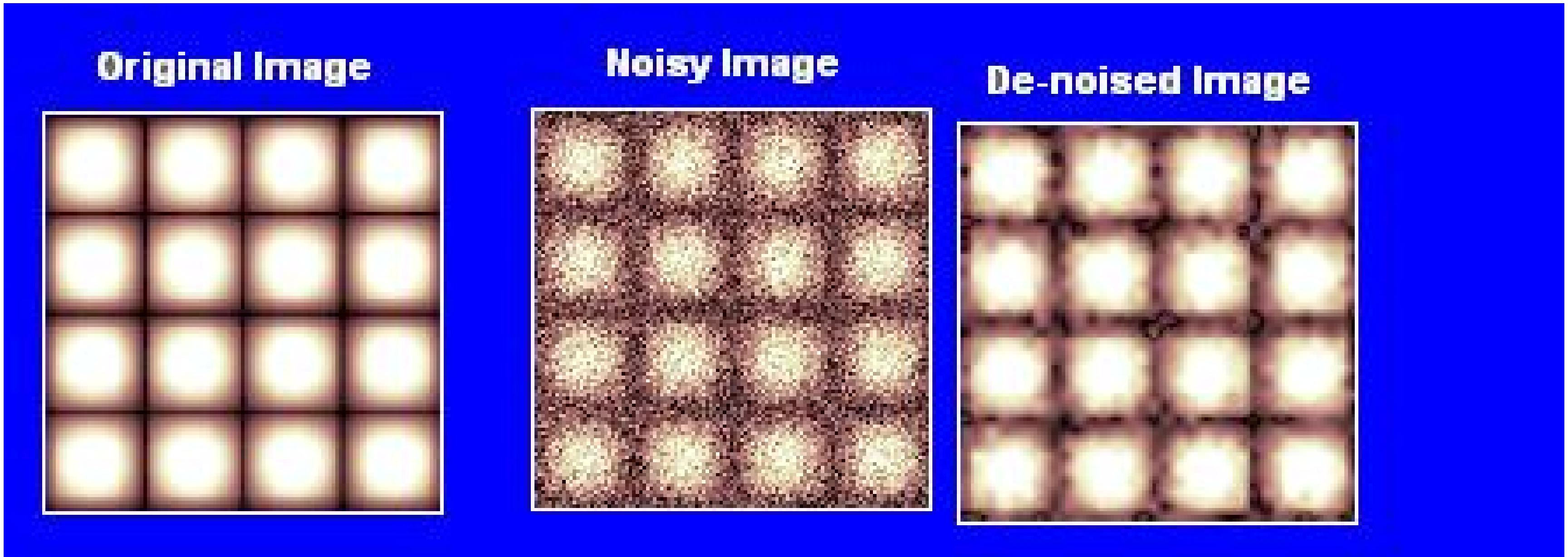
- Highest Frequencies Appear at the Start of The Original Signal
- Approximations Appear Less and Less Noisy
- Also Lose Progressively More High-frequency Information.
- In $A_5$, About the First 20% of the Signal is Truncated




| Example Analysis |
| --- |
| Noisy Doppler |
| **MAT-file** |
| noisdopp.mat |
| **Wavelet** |
| sym4 |
| **Level** |
| 5 |

# Another De-Noising



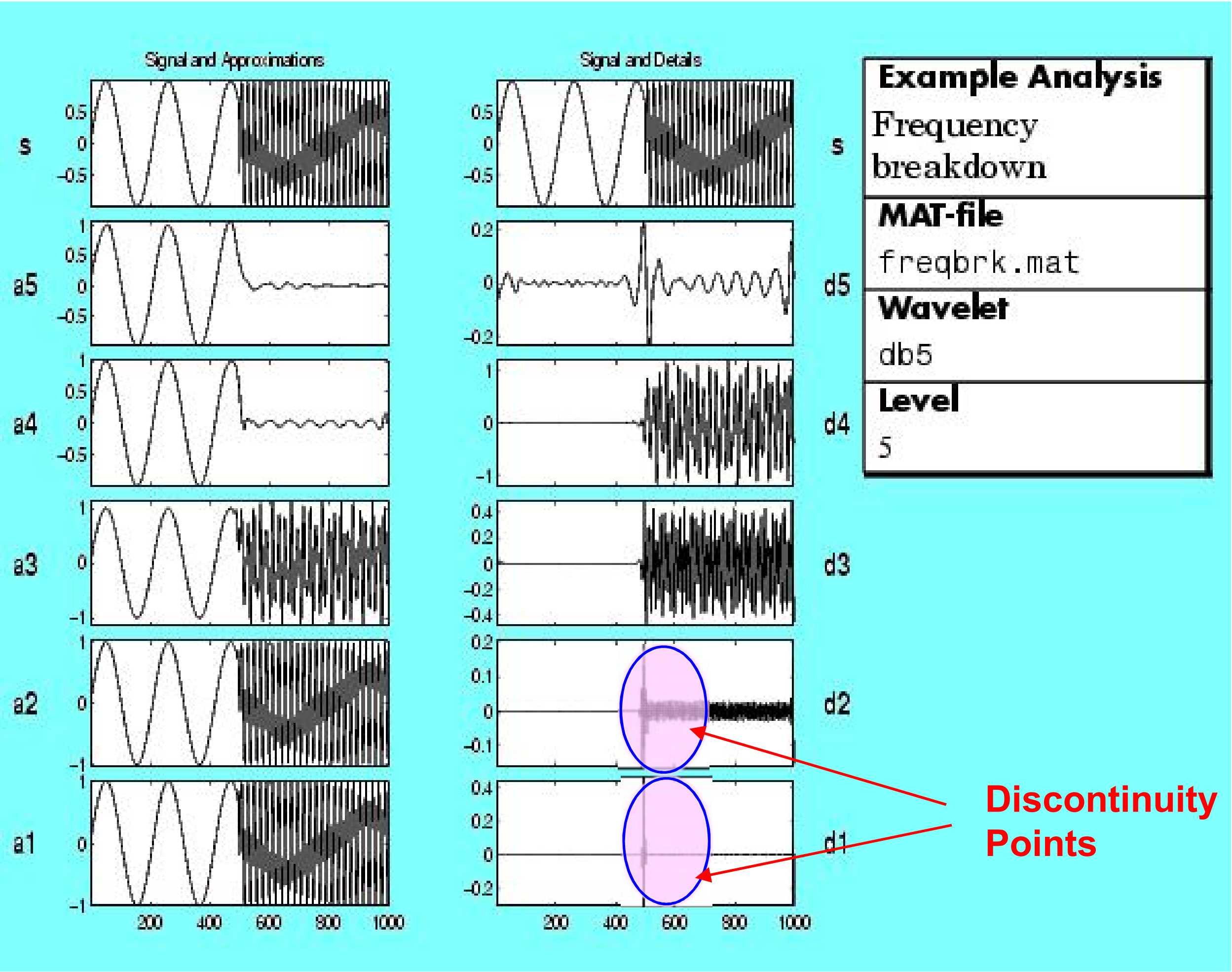

```
% Use wdencmp for image de-noising.

    % find default values (see ddencmp).
    [thr,sorh,keepapp] = ddencmp('den','wv',x);

    % denoise image using global thresholding option.
    xd = wdencmp('gbl',x,'sym4',2,thr,sorh,keepapp);
```

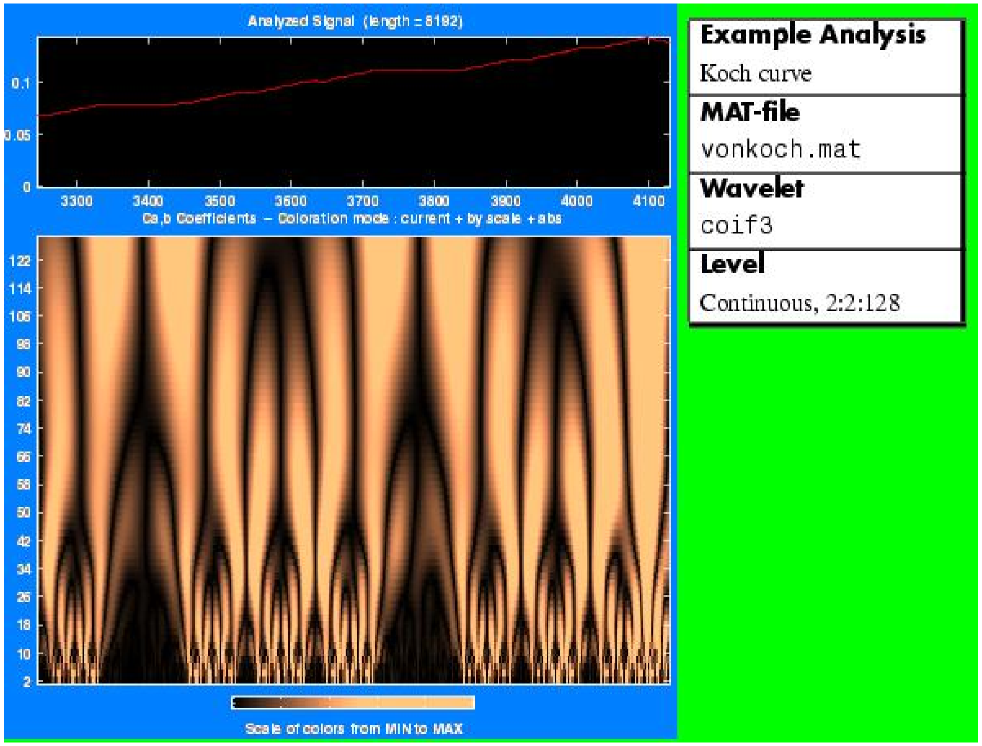
# Detecting Discontinuities and Breakdown Points

- The Discontinuous Signal Consists of a Slow Sine Wave Abruptly Followed by a Medium Sine Wave.
- The 1st and 2nd Level Details ($D_1$ and $D_2$) Show the Discontinuity Most Clearly
- Things to be Detected
  - The site of the change
  - The type of change (a rupture of the signal, or an abrupt change in its first or second derivative)
  - The amplitude of the change



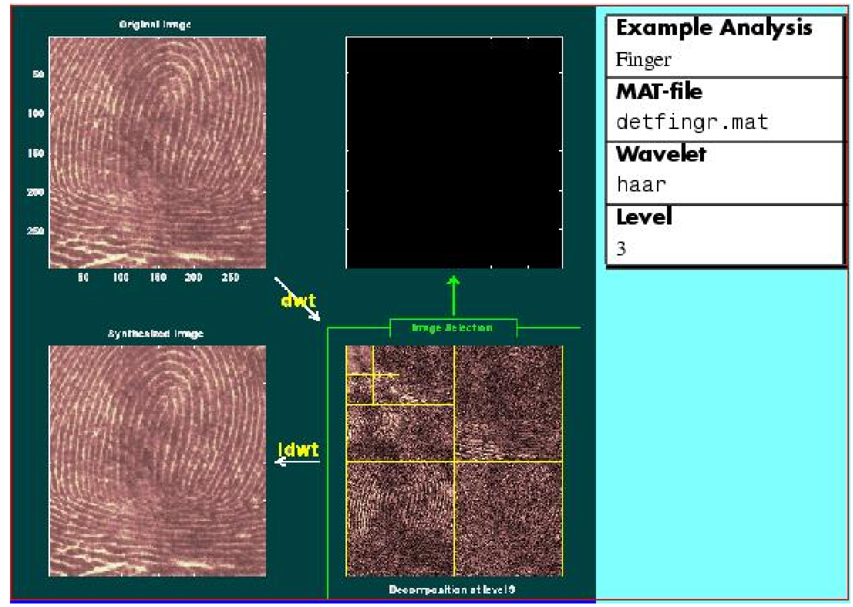
# Detecting Self-Similarity

- Purpose
  - How analysis by wavelets can detect a self-similar, or fractal, signal.
  - The signal here is the Koch curve -- a synthetic signal that is built recursively
- Analysis
  - If a signal is similar to itself at different scales, then the "resemblance index" or wavelet coefficients also will be similar at different scales.
  - In the coefficients plot, which shows scale on the vertical axis, this self-similarity generates a characteristic pattern.



# Compressing Images

- Fingerprints
  - FBI maintains a large database of fingerprints — about 30 million sets of them.
  - The cost of storing all this data runs to hundreds of millions of dollars.
- Results
  - Values under the threshold are forced to zero, achieving about 42% zeros while retaining almost all (99.96%) the energy of the original image.
  - By turning to wavelets, the FBI has achieved a 15:1 compression ratio
  - better than the more traditional JPEG compression




# Identifying Pure Frequencies

- Purpose
  - ➢Resolving a signal into constituent sinusoids of different frequencies
  - ➢The signal is a sum of three pure sine waves
- Analysis
  - ➢D1 contains signal components whose period is between 1 and 2.
  - ➢Zooming in on detail D1 reveals that each "belly" is composed of 10 oscillations.
  - ➢D3 and D4 contain the medium sine frequencies.
  - ➢There is a breakdown between approximations A3 and A4 -> The medium frequency been subtracted.
  - ➢Approximations A1 to A3 be used to estimate the medium sine.
  - ➢Zooming in on A1 reveals a period of around 20.