SEB4233
Biomedical Signal Processing

Z-Transform

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Introduction

- The *z*-transform is the discrete-time counterpart of the Laplace transform.
- It can be used to assess the characteristic of discrete-time systems in terms of its impulse response and frequency response.
- The *z*-transform can be used determine the solution to the difference equation.

Definition of *Z*-Transform

- For a given sequence $x[n]$, its z-transform X(z) is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(n) = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

Z-Transform Example

A signal is defined as

$$x(n) = a^n \qquad n \geq 0$$
$$= 0 \qquad \text{elsewhere}$$

$$X(z) = \sum_{n=0}^{\infty} (a)^n z^{-n} = 1 + a^1 z^{-1} + a^2 z^{-2} + a^3 z^{-3} + a^4 z^{-4} + ...$$

(Open form)

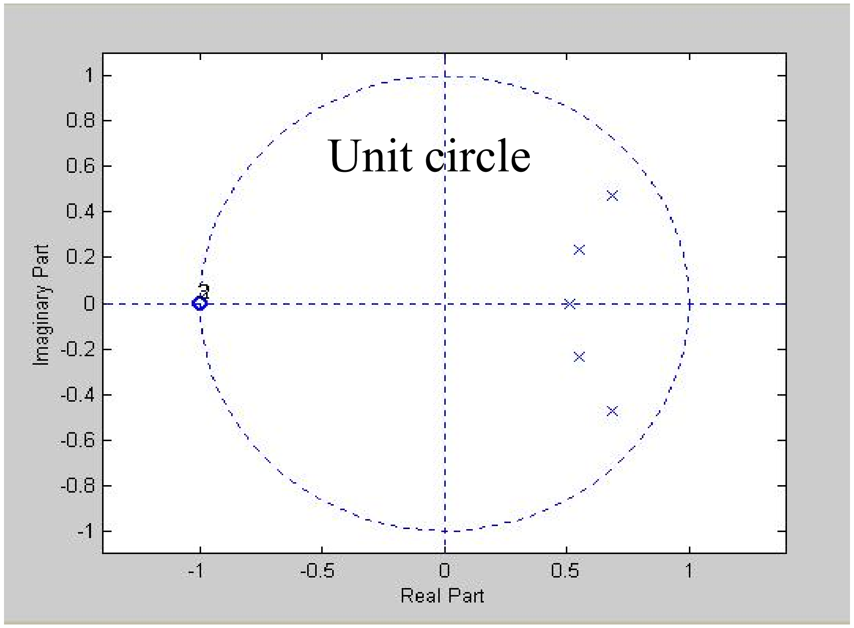
$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

(Close form)

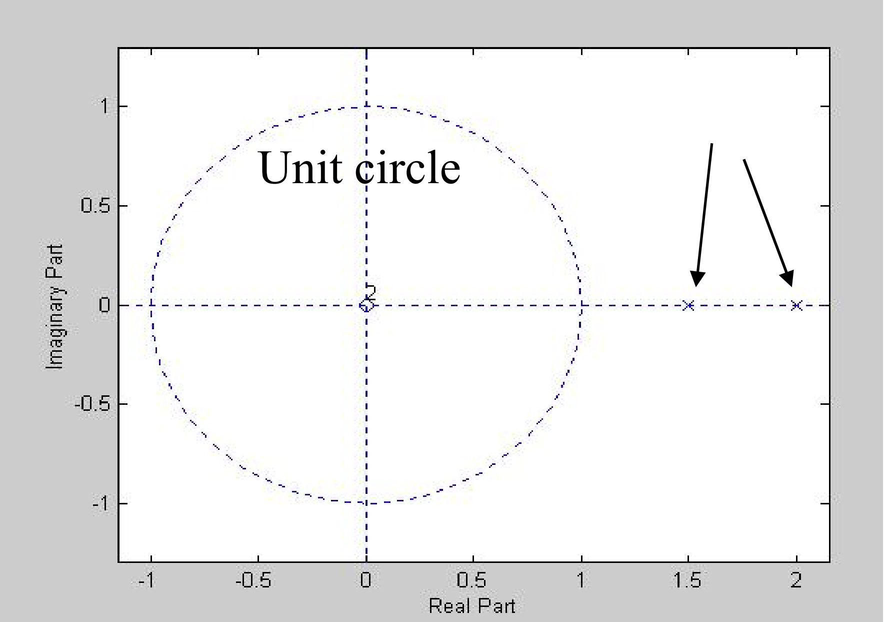
From the close form solution, there is a pole where $z = a$ and a zero.

Z-Plane and Stability

- All the possible values of $X(z)$ lies in the z-plane.
- The maximum are at the poles and the zero value are at the zeros.
- For causal sequence, the system is stable if the poles are in the unit circle.



(a) Stable System



(b) Unstable system

Solution for Difference Equation and Transfer Function

- The z-transform can be used to determine the solution to difference equation. Given that the input-output relationship of a linear time-invariant system is as follows

$$y(n) = a(1)y(n\text{-}1) + a(2)y(n\text{-}2) = b(0)x(n) + b(1)x(n\text{-}1) + b(2)x(n\text{-}2)$$

- The z-transform is

$$Y(z) + a(1)Y(z)\,z^{-1} + a(2)Y(z)\,z^{-2} = b(0)X(z) + b(1)X(z)\,z^{-1} + b(2)X(z)\,z^{-2}$$

$$H(z) = \frac{Y(z)}{X(z)} = \left[\frac{b(0) + b(1)z^{-1} + b(2)z^{-2}}{1 + a(1))z^{-1} + a(2)z^{-2}}\right]$$

where $H(z)$ is the transfer function.

General Form of the Transfer Function

- For more general case, the transfer function is in the form

$$H(z) = \frac{\sum_{n=0}^{N} b(n) z^{-n}}{\sum_{n=0}^{N} a(n) z^{-n}}$$

where N is the polynomial order. The transfer function when factorized in term of the roots is

$$H(z) = \frac{\prod_{n=0}^{N} (1 - \beta(n) z^{-n})}{\prod_{n=0}^{N} (1 - \alpha(n) z^{-n})}$$

Inverse Z-Transform

- The system impulse response $h(n)$ is obtained from $H(z)$ by taking the inverse z-transform. If the following transfer function is used as example

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 + a(1)z^{-1} + a(2)z^{-2}}$$

$$= \frac{1}{(1 - \alpha(1)z^{-1})(1 - \alpha(2)z^{-1})}$$

- The application of the partial fraction expansion results in

$$H(z) = \frac{1}{(1-\alpha(1)z^{-1})(1-\alpha(2)z^{-1})} = \frac{A_0}{1-\alpha(1)z^{-1}} + \frac{A_1}{1-\alpha(2)z^{-1}}$$

Relationship between the *Z*-Transform and Fourier Transform

- The Fourier transform can be obtained from the z-transform by making the substitution $z = \exp(j2\pi f)$
- A transfer function is

$$H(z) = \frac{1}{1 - z^{-1} + \frac{1}{2} z^{-2}}$$

- The Fourier transform is

$$H(\exp(j2\pi f)) = \frac{1}{1 - \exp(-j2\pi f) + \frac{1}{2}\exp(-j4\pi f)} = \frac{1}{1 - \cos(2\pi f) + \frac{1}{2}\cos(4\pi f) + j\left(\sin(2\pi f) - \frac{1}{2}\sin(4\pi f)\right)}$$

Relationship between the *Z*-Transform and Fourier Transform

- The Fourier transform when defined in terms of the magnitude and phase is

$$|H(\exp(j2\pi f))| = \frac{1}{\sqrt{\left(1-\cos(2\pi f)+\frac{1}{2}\cos(4\pi f)\right)^2+\left(\sin(2\pi f)-\frac{1}{2}\sin(4\pi f)\right)^2}}$$

$$\phi(\exp(j2\pi f)) = -\tan^{-1}\left[\frac{\left(\sin(2\pi f)-\frac{1}{2}\sin(4\pi f)\right)}{\left(1-\cos(2\pi f)+\frac{1}{2}\cos(4\pi f)\right)}\right]$$

Frequency Response of Transfer Function