SKMM 3033
Finite Element Method

Topic 5: Equilibrium equations using energy approach

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By the end of the notes:

The students are expected:

- To understand the Minimum Potential Energy theorem used to solve FEM problems

Minimum Potential Energy Theorem

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F$$

The theorem states that:

"…Of all the displacements that satisfy the boundary conditions of a structural system, those corresponding to equilibrium configurations make the total potential energy to assume a minimum value…"

$$\frac{d\Pi}{dQ} = 0$$

Minimum Potential Energy Theorem

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F$$

Assuming that we have 2 element, 3-noded finite element model:

$$Q = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}; K = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix}; F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

Minimum Potential Energy Theorem

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F$$

Assuming that we have 2 element, 3-noded finite element model:

$$\Pi = \frac{1}{2}\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}^T \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix}\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} - \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}^T \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} Q_1 & Q_2 & Q_3 \end{pmatrix}\begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix}\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} - \begin{pmatrix} Q_1 & Q_2 & Q_3 \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} Q_1\left(Q_1 k_{11} + Q_2 k_{21} + Q_3 k_{31}\right) \\ +Q_2\left(Q_1 k_{12} + Q_2 k_{22} + Q_3 k_{32}\right) \\ +Q_3\left(Q_1 k_{13} + Q_2 k_{23} + Q_3 k_{33}\right) \end{bmatrix} - \left(Q_1 F_1 + Q_2 F_2 + Q_3 F_3\right)$$

$$= \frac{1}{2}\begin{bmatrix} \left(Q_1 k_{11} Q_1 + Q_2 k_{21} Q_1 + Q_3 k_{31} Q_1\right) \\ +\left(Q_1 k_{12} Q_2 + Q_2 k_{22} Q_2 + Q_3 k_{32} Q_2\right) \\ +\left(Q_1 k_{13} Q_3 + Q_2 k_{23} Q_3 + Q_3 k_{33} Q_3\right) \end{bmatrix} - \left(Q_1 F_1 + Q_2 F_2 + Q_3 F_3\right)$$

Minimum Potential Energy Theorem

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Assuming that we have 2 element, 3-noded finite element model:

$$\Pi = \frac{1}{2}\begin{bmatrix} \left(Q_1 k_{11} Q_1 + Q_2 k_{21} Q_1 + Q_3 k_{31} Q_1\right) \\ +\left(Q_1 k_{12} Q_2 + Q_2 k_{22} Q_2 + Q_3 k_{32} Q_2\right) \\ +\left(Q_1 k_{13} Q_3 + Q_2 k_{23} Q_3 + Q_3 k_{33} Q_3\right) \end{bmatrix} - \left(Q_1 F_1 + Q_2 F_2 + Q_3 F_3\right)$$

Introducing Boundary Condition

$$Q_1 = a_1$$

$$= \frac{1}{2}\begin{bmatrix} \left(a_1 k_{11} a_1 + Q_2 k_{21} a_1 + Q_3 k_{31} a_1\right) \\ +\left(a_1 k_{12} Q_2 + Q_2 k_{22} Q_2 + Q_3 k_{32} Q_2\right) \\ +\left(a_1 k_{13} Q_3 + Q_2 k_{23} Q_3 + Q_3 k_{33} Q_3\right) \end{bmatrix} - \left(Q_1 F_1 + Q_2 F_2 + Q_3 F_3\right)$$

Minimum Potential Energy Theorem

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F$$

Assuming that we have 2 element, 3-noded finite element model:

$$\Pi = \frac{1}{2}\begin{bmatrix} (a_1 k_{11} a_1 + Q_2 k_{21} a_1 + Q_3 k_{31} a_1) \\ +(a_1 k_{12} Q_2 + Q_2 k_{22} Q_2 + Q_3 k_{32} Q_2) \\ +(a_1 k_{13} Q_3 + Q_2 k_{23} Q_3 + Q_3 k_{33} Q_3) \end{bmatrix} - (a_1 F_1 + Q_2 F_2 + Q_3 F_3)$$

Introducing Boundary Condition

$$Q_1 = a_1$$

$$\frac{d\Pi}{dQ_i} = 0; i = 2,3$$

$$\frac{d\Pi}{dQ_2} = \frac{1}{2}\begin{bmatrix} (k_{21} a_1) \\ +(a_1 k_{12} + 2k_{22} Q_2 + Q_3 k_{32}) \\ +(k_{23} Q_3) \end{bmatrix} - (F_2) = 0$$

Why ???

$$\because k_{12} = k_{21}$$
$$\because k_{23} = k_{32}$$

$$\therefore k_{22} Q_2 + k_{23} Q_3 = F_2 - k_{21} a_1$$

Minimum Potential Energy Theorem

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F$$

Assuming that we have 2 element, 3-noded finite element model:

$$\Pi = \frac{1}{2}\begin{bmatrix} (a_1 k_{11} a_1 + Q_2 k_{21} a_1 + Q_3 k_{31} a_1) \\ +(a_1 k_{12} Q_2 + Q_2 k_{22} Q_2 + Q_3 k_{32} Q_2) \\ +(a_1 k_{13} Q_3 + Q_2 k_{23} Q_3 + Q_3 k_{33} Q_3) \end{bmatrix} - (a_1 F_1 + Q_2 F_2 + Q_3 F_3)$$

Introducing Boundary Condition

$$Q_1 = a_1$$

$$\frac{d\Pi}{dQ_i} = 0; i = 2,3$$

$$\frac{d\Pi}{dQ_3} = \frac{1}{2}\begin{bmatrix} (k_{31} a_1) \\ +(k_{32} Q_2) \\ +(a_1 k_{13} + Q_2 k_{23} + 2 k_{33} Q_3) \end{bmatrix} - (F_3) = 0$$

$$\because k_{13} = k_{31}$$

$$\because k_{23} = k_{32}$$

$$\therefore k_{32} Q_2 + k_{33} Q_3 = F_3 - k_{31} a_1$$

Minimum Potential Energy Theorem

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F$$

Assuming that we have 2 element, 3-noded finite element model:

Introducing Boundary Condition

$$Q_1 = a_1$$

$$\frac{d\Pi}{dQ_i} = 0; i = 2,3$$

$$k_{22}Q_2 + k_{23}Q_3 = F_2 - k_{21}a_1$$
$$k_{32}Q_2 + k_{33}Q_3 = F_3 - k_{31}a_1$$

$$\therefore \begin{pmatrix} k_{22} & k_{23} \\ k_{32} & k_{33} \end{pmatrix} \begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} F_2 - k_{21}a_1 \\ F_3 - k_{31}a_1 \end{pmatrix}$$

Minimum Potential Energy Theorem

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F$$

Assuming that we have 2 element, 3-noded finite element model:

Introducing Boundary Condition

$$Q_1 = a_1$$

$$\frac{d\Pi}{dQ_i} = 0; i = 2,3$$

$$\therefore KQ = F$$

$$\therefore \begin{pmatrix} k_{22} & k_{23} \\ k_{32} & k_{33} \end{pmatrix} \begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} F_2 - k_{21} a_1 \\ F_3 - k_{31} a_1 \end{pmatrix}$$

Minimum Potential Energy Theorem

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F$$

Assuming that we have 2 element, 3-noded finite element model:

Introducing Boundary Condition

$$Q_1 = a_1$$

$$\frac{d\Pi}{dQ_i} = 0; i = 2,3$$

To simplify the problem, let's assume node 1 is constrained

$$Q_1 = a_1 = 0$$

$$\therefore \begin{pmatrix} k_{22} & k_{23} \\ k_{32} & k_{33} \end{pmatrix} \begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} F_2 \\ F_3 \end{pmatrix}$$

Minimum Potential Energy Theorem

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F$$

The method is also called elimination method

Introducing Boundary Condition

$$Q_1 = a_1 = 0$$

$$KQ = F$$

$$\begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

Minimum Potential Energy Theorem

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F$$

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Introducing Boundary Condition

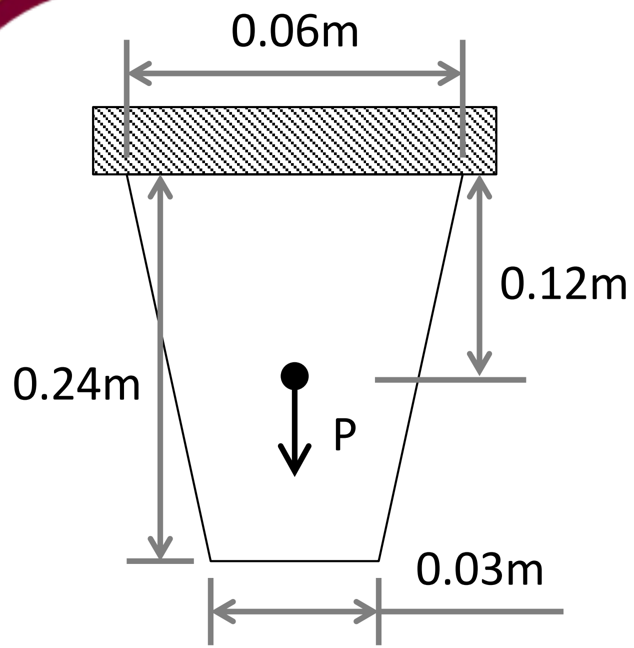
$$Q_1 = a_1 = 0$$

$$KQ = F$$

$$\begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

$$\therefore \begin{pmatrix} k_{22} & k_{23} \\ k_{32} & k_{33} \end{pmatrix} \begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} F_2 \\ F_3 \end{pmatrix}$$

Example

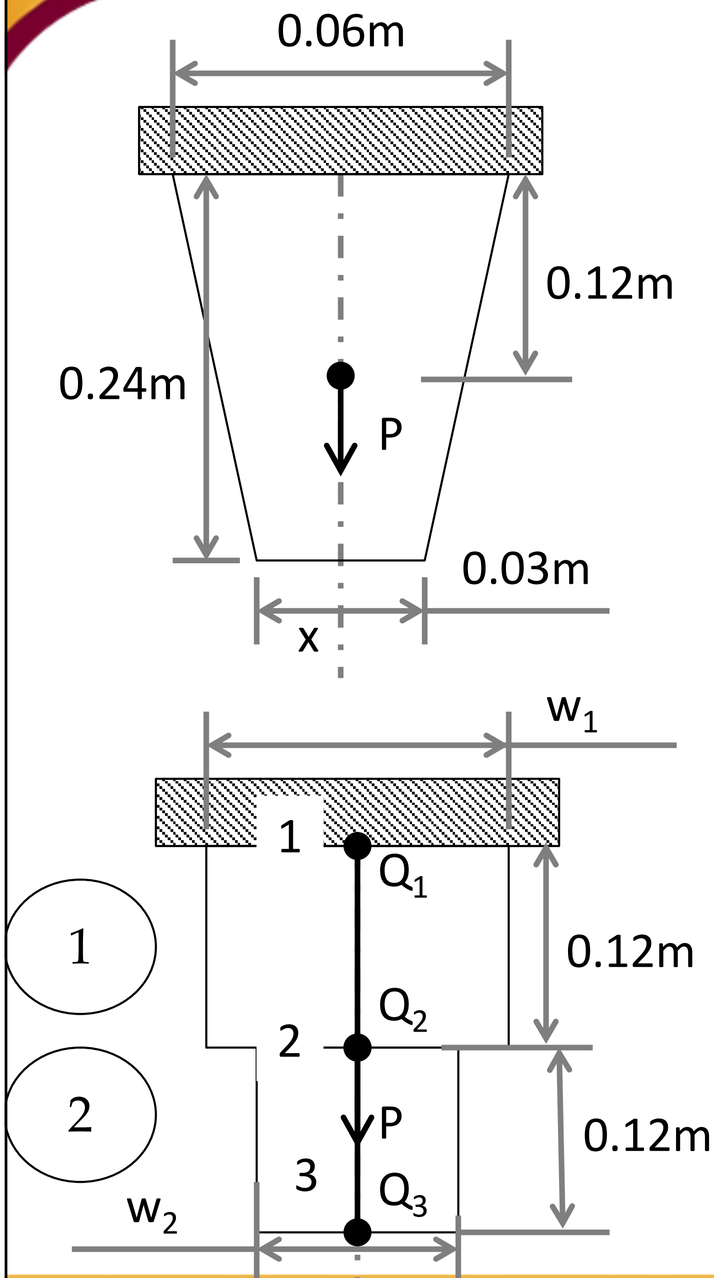


The plate given the the figure has a uniform thickness of 1m, Modulus elasticity of 210GPa and a density of 8,000 kg/m^3. The plate is subjected to a point load, P=100N at its midpoint and also to its self-weight.

Assuming the plate consists of two elements, calculate:

- The global displacement vector, Q

Example



The plate given the the figure has a uniform thickness of 1m, Modulus elasticity of 210GPa and a density of 8,000 kg/m³. The plate is subjected to a point load, P=100N at its midpoint and also to its self-weight.

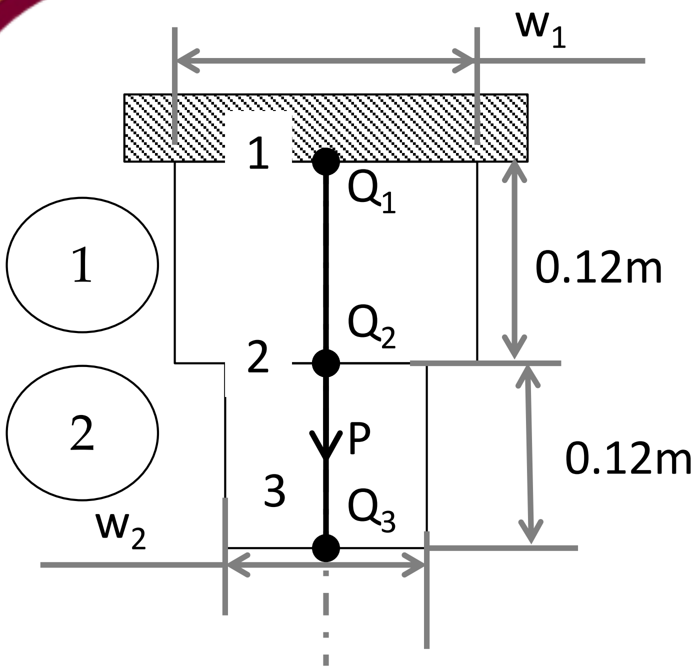
The global load vector, F

$$F = \begin{pmatrix} 247.212 \\ 523.792 \\ 176.58 \end{pmatrix} N$$

The structural stiffness matrix, K

$$K = \frac{210\times10^{9}}{0.12}\begin{pmatrix} 0.0525 & -0.0525 & 0 \\ -0.0525 & 0.09 & -0.0375 \\ 0 & -0.0375 & 0.0375 \end{pmatrix}$$

Example



Rigid Support

$$Q_1 = 0$$

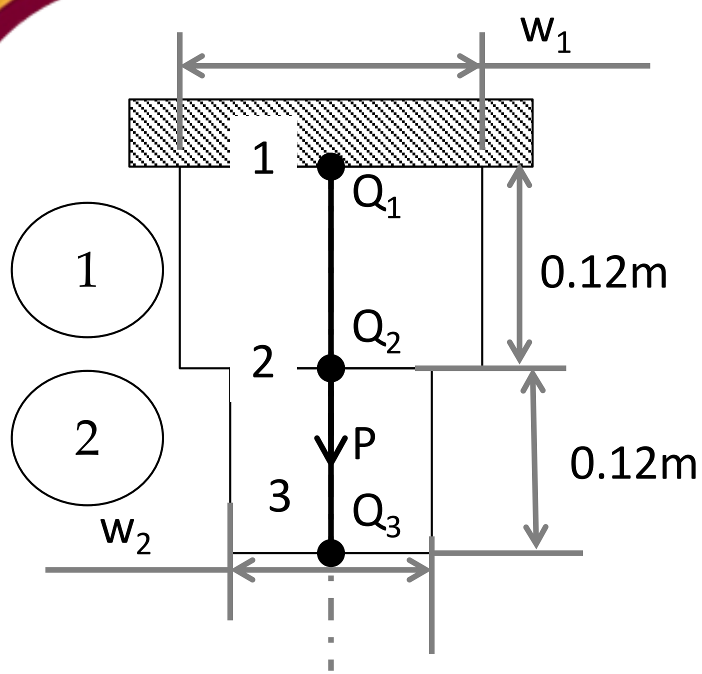
The plate given the the figure has a uniform thickness of 1m, Modulus elasticity of 210GPa and a density of 8,000 kg/m³. The plate is subjected to a point load, P=100N at its midpoint and also to its self-weight.

The global displacement vector, Q
(Using the elimination method)

$$KQ = F$$

$$\frac{210 \times 10^9}{0.12}\begin{pmatrix} 0.0525 & -0.0525 & 0 \\ -0.0525 & 0.09 & -0.0375 \\ 0 & -0.0375 & 0.0375 \end{pmatrix}\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} 247.212 \\ 523.792 \\ 176.58 \end{pmatrix}$$

Example



Rigid Support

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The plate given the the figure has a uniform thickness of 1m, Modulus elasticity of 210GPa and a density of 8,000 kg/m³. The plate is subjected to a point load, P=100N at its midpoint and also to its self-weight.

The global displacement vector, Q
(Using the elimination method)

$$\frac{210\times10^9}{0.12}\begin{pmatrix} 0.09 & -0.0375 \\ -0.0375 & 0.0375 \end{pmatrix}\begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} 523.792 \\ 176.58 \end{pmatrix}$$

$$\therefore \begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} 7.623 \\ 10.314 \end{pmatrix} nm$$

By the end of the notes:

You are expected to be able:

- To understand the Minimum Potential Energy theorem used to solve FEM problems