SKMM 3033
Finite Element Method

Topic 4: Potential energy approach for 1-D bar stiffness matrix

DR WILLIAM CHONG WOEI FONG
Department of Aeronautics, Automotive and Ocean Engineering
Faculty of Mechanical Engineering

By the end of the notes:

The students are expected:

- To formulate the element stiffness matrix for a 1-D bar element using the potential energy approach

Stiffness Matrix, k

Potential Energy Approach: $$\Pi = U + \Omega$$

Strain Energy | Work Potential

<u>For non-uniform bar:</u>

$$\Pi = \frac{1}{2}\int_L \sigma^T \varepsilon A\,dx \boxed{- \int_L u^T f A\,dx - \int_L u^T T\,dx - \sum_i u_i p_i}$$

Stiffness Matrix, k

Potential Energy Approach: $\Pi = U + \Omega$

For the bar discretised into finite elements:

$$\Pi = \sum_e \frac{1}{2}\int_e \sigma^T \varepsilon A\,dx - \sum_e \int_e u^T fA\,dx - \sum_e \int_e u^T T\,dx - \sum_i Q_i P_i$$

$$= \sum_e U_e - \sum_e \int_e u^T fA\,dx - \sum_e \int_e u^T T\,dx - \sum_i Q_i P_i$$

Where:

$$\boxed{U_e = \frac{1}{2}\int_e \sigma^T \varepsilon A\,dx}$$

Element Internal Strain Energy

Stiffness Matrix, k

Potential Energy Approach:

Element Internal Strain Energy:

$$U_e = \frac{1}{2}\int_e \sigma^T \varepsilon A\, dx$$

$$\varepsilon = \{B\}\{q\}$$

$$\sigma = E\{B\}\{q\}$$

$$U_e = \frac{1}{2}\int_e \left[E\{B\}\{q\}\right]^T \{B\}\{q\} A_e\, dx$$

$$= \frac{1}{2}\int_e E\{q\}^T \{B\}^T \{B\}\{q\} A_e\, dx$$

$$= \frac{1}{2}\{q\}^T \int_e E\left[\{B\}^T \{B\} A_e\right] dx \{q\}$$

$$\frac{d\xi}{dx} = 2\left(\frac{1}{x_2 - x_1}\right)$$

$$\therefore dx = \left(\frac{x_2 - x_1}{2}\right) d\xi = \left(\frac{l_e}{2}\right) d\xi$$

Stiffness Matrix, k

Potential Energy Approach:

Element Internal Strain Energy:

$$U_e = \frac{1}{2}\int_e \sigma^T \varepsilon A\,dx$$

$$\varepsilon = \{B\}\{q\}$$

$$\sigma = E\{B\}\{q\}$$

$$U_e = \frac{1}{2}\{q\}^T \int_e E\left[\{B\}^T\{B\}A_e\left(\frac{l_e}{2}\right)d\xi\right]\{q\}$$

$$= \frac{1}{2}\{q\}^T\left[E\{B\}^T\{B\}A_e\left(\frac{l_e}{2}\right)\int_{-1}^{1} d\xi\right]\{q\}$$

$$\int_{-1}^{1} d\xi = 2$$

Stiffness Matrix, k

Potential Energy Approach:

Element Internal Strain Energy:

$$U_e = \frac{1}{2}\int_e \sigma^T \varepsilon A\, dx$$

$$\varepsilon = \{B\}\{q\}$$

$$\sigma = E\{B\}\{q\}$$

$$\{B\} = \frac{1}{l_e}\begin{pmatrix} -1 & 1 \end{pmatrix}$$

$$U_e = \frac{1}{2}\{q\}^T\left[E\{B\}^T\{B\}A_e l_e\right]\{q\}$$

$$= \frac{1}{2}\{q\}^T\left[EA_e l_e \frac{1}{l_e}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\frac{1}{l_e}\begin{pmatrix} -1 & 1 \end{pmatrix}\right]\{q\}$$

$$= \frac{1}{2}\{q\}^T\left[EA_e l_e \frac{1}{l_e^2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \end{pmatrix}\right]\{q\}$$

$$= \frac{1}{2}\{q\}^T \frac{EA_e}{l_e}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\{q\}$$

Stiffness Matrix, k

Potential Energy Approach:

Element Internal Strain Energy:

$$U_e = \frac{1}{2}\int_e \sigma^T \varepsilon A\,dx$$

$$\varepsilon = \{B\}\{q\}$$

$$\sigma = E\{B\}\{q\}$$

$$U_e = \frac{1}{2}\{q\}^T EA_e \frac{1}{l_e}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\{q\}$$

$$= \frac{1}{2}\{q\}^T k^e \{q\}$$

$$\boxed{k^e = \frac{EA_e}{l_e}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}$$

E – Modulus of elasticity
A_e – Cross sectional area
l_e –Element length

Element Force Terms

Traction Force, T

$$\int_e u^T T \, dx$$

$$u = \begin{pmatrix} N_1 & N_2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \{N\}\{q\}$$

$$N_1 = \frac{(1-\xi)}{2} \rightarrow \int_{-1}^{1} \frac{(1-\xi)}{2} d\xi = 1$$

$$N_2 = \frac{(\xi+1)}{2} \rightarrow \int_{-1}^{1} \frac{(\xi+1)}{2} d\xi = 1$$

$$\int_e u^T T \, dx$$

$$= \{q\}^T \frac{Tl_e}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \{q\}^T T^e$$

$$\boxed{\therefore T^e = \frac{Tl_e}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

Element Force Terms

Body Force, f

$$\int_e u^T fA\,dx$$

$$u = \begin{pmatrix} N_1 & N_2 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \{N\}\{q\}$$

$$N_1 = \frac{(1-\xi)}{2} \rightarrow \int_{-1}^{1} \frac{(1-\xi)}{2} d\xi = 1$$

$$N_2 = \frac{(\xi+1)}{2} \rightarrow \int_{-1}^{1} \frac{(\xi+1)}{2} d\xi = 1$$

$$\int_e u^T fA\,dx = \int_e [\{N\}\{q\}]^T fAdx$$

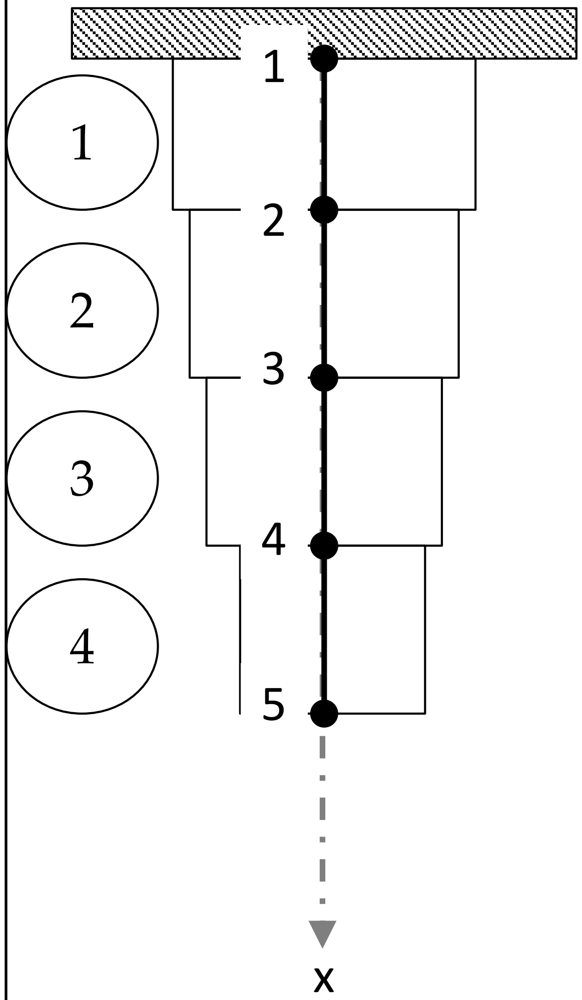
$$\Downarrow$$

Try to derive it yourself !!!

$$\boxed{\therefore f^e = \frac{fA_e l_e}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

Assembly of Global Stiffness Matrix and Load Vector

From element to global



Element Connectivity Table

Nodes	
1	2
2	3
3	4
4	5

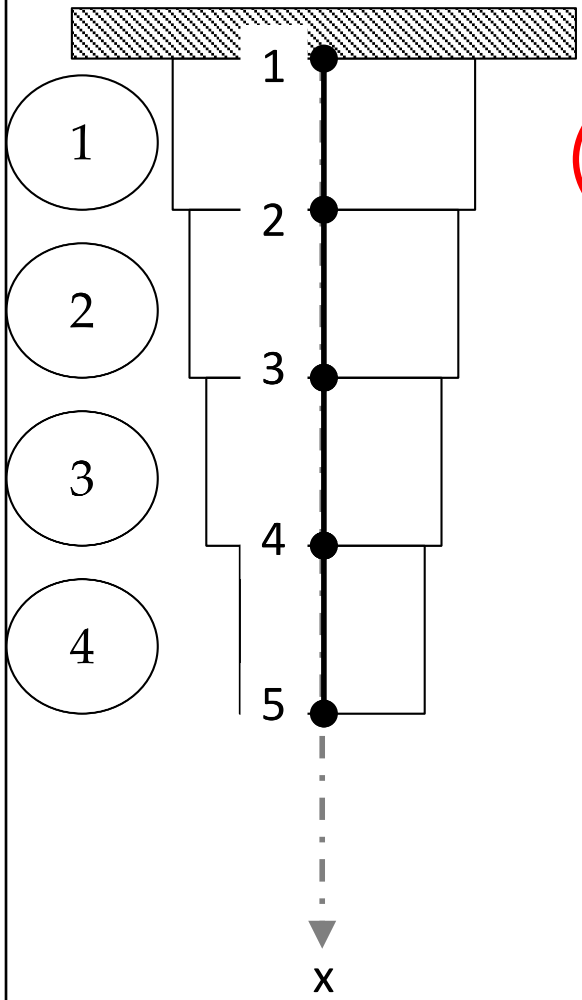
$$\Pi = \frac{1}{2} Q^T K Q - Q^T F$$

For an FE model with 4 elements:

- 5 nodes
- 5x5 Global Stiffness Matrix, K

$$K = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} \end{pmatrix} \qquad Q = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{pmatrix}$$

Global Stiffness Matrix, K



Element Connectivity Table

Nodes	
1	2
2	3
3	4
4	5

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F$$

What is the stiffness matrix for element 1??

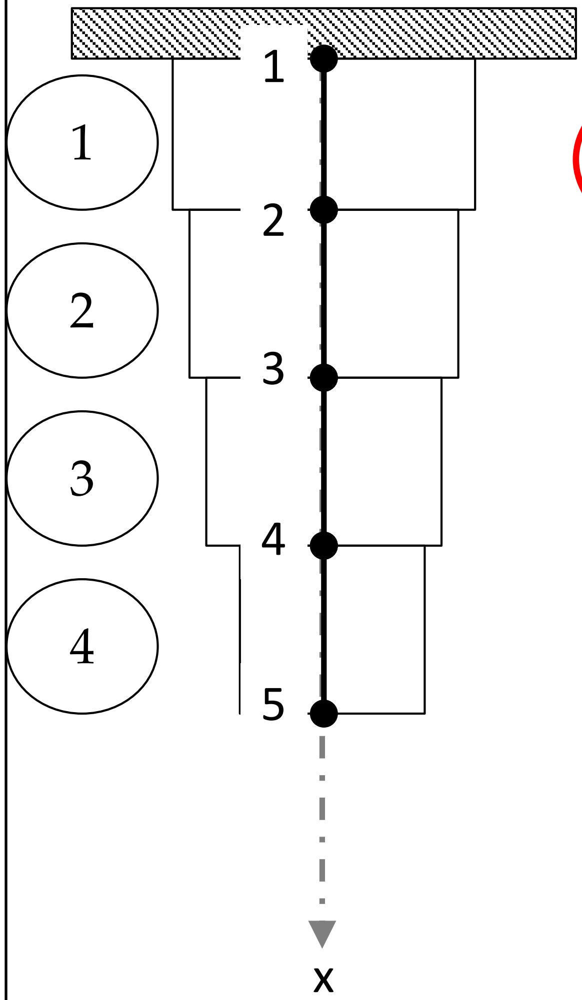
Element Stiffness Vector (Element 1)

$$\{q\} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

$$k^1 = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array}$$

$$= \frac{EA_1}{l_1} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Global Stiffness Matrix, K



Element Connectivity Table

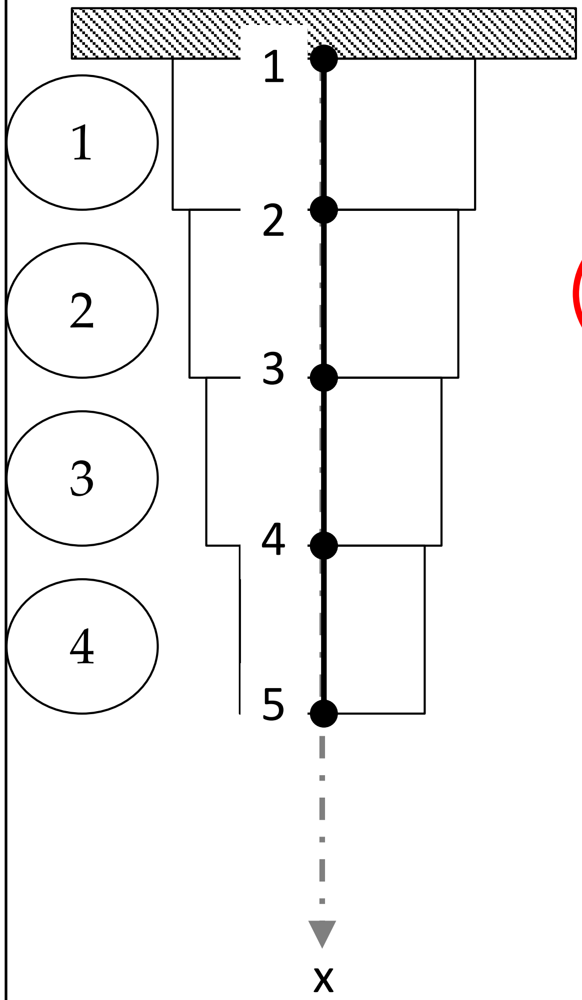
Nodes	
1	2
2	3
3	4
4	5

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F$$

What is the stiffness matrix for element 1??

$$k_1 = \frac{EA_1}{l_1} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Global Stiffness Matrix, K



Element Connectivity Table

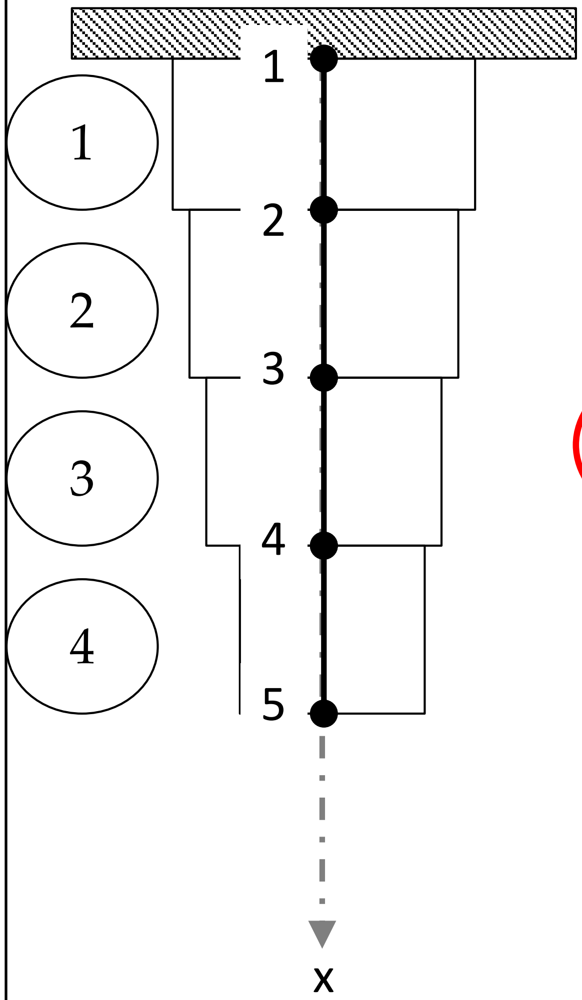
	Nodes	
	1	2
	2	3
	3	4
	4	5

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F$$

What is the stiffness matrix for element 2??

$$k_2 = \frac{EA_2}{l_2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Global Stiffness Matrix, K



Element Connectivity Table

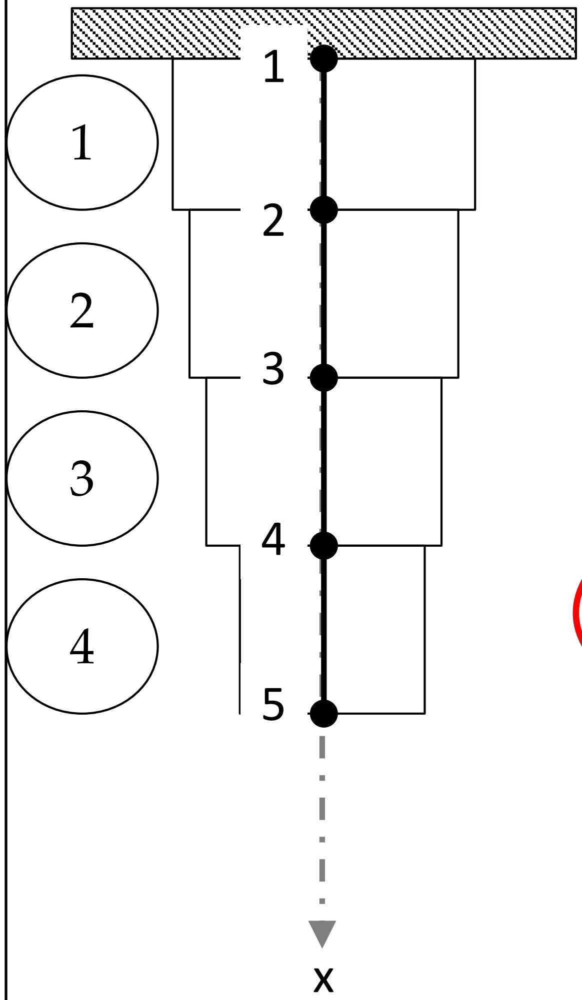
Nodes	
1	2
2	3
3	4
4	5

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F$$

What is the stiffness matrix for element 3??

$$k_3 = \frac{E_3 A_3}{l_3} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Global Stiffness Matrix, K



Element Connectivity Table

Nodes	
1	2
2	3
3	4
4	5

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F$$

What is the stiffness matrix for element 4??

$$k_4 = \frac{E_4 A_4}{l_4} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Global Stiffness Matrix, K

$$K = k_1 + k_2 + k_3 + k_4$$

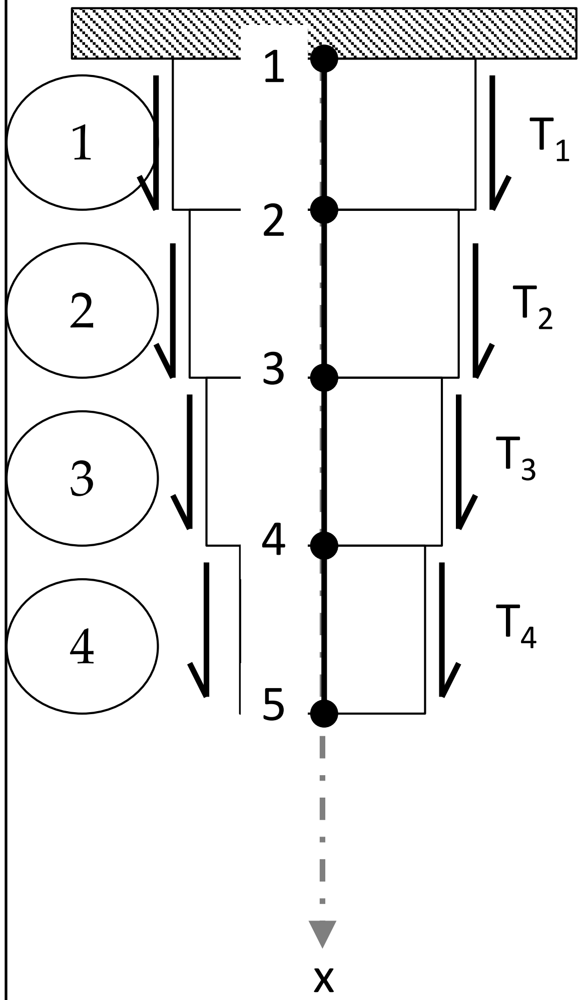
$$= \frac{E_1 A_1}{l_1}\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \frac{E_2 A_2}{l_2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ \frac{E_3 A_3}{l_3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \frac{E_4 A_4}{l_4}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Global Stiffness Matrix, K

$$K = \begin{pmatrix} \frac{E_1A_1}{l_1} & -\frac{E_1A_1}{l_1} & 0 & 0 & 0 \\ -\frac{E_1A_1}{l_1} & \frac{E_1A_1}{l_1}+\frac{E_2A_2}{l_2} & -\frac{E_2A_2}{l_2} & 0 & 0 \\ 0 & -\frac{E_2A_2}{l_2} & \frac{E_2A_2}{l_2}+\frac{E_3A_3}{l_3} & -\frac{E_3A_3}{l_3} & 0 \\ 0 & 0 & -\frac{E_3A_3}{l_3} & \frac{E_3A_3}{l_3}+\frac{E_4A_4}{l_4} & -\frac{E_4A_4}{l_4} \\ 0 & 0 & 0 & -\frac{E_4A_4}{l_4} & \frac{E_4A_4}{l_4} \end{pmatrix}$$

Global Load Vector



Element Connectivity Table

	Nodes
1	2
2	3
3	4
4	5

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F$$

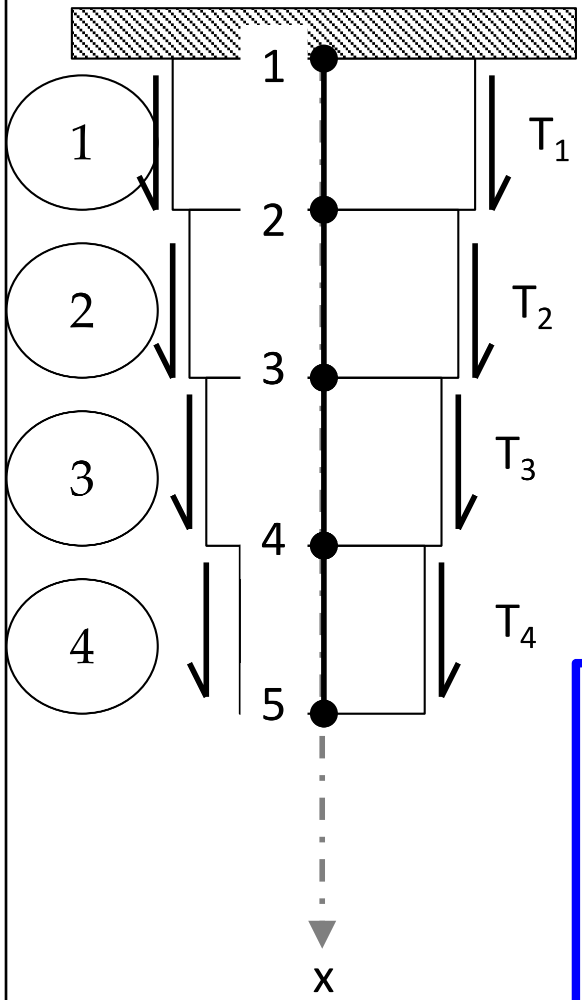
For an FE model with 4 elements:

- 5 nodes
- 5x1 Global Load Vector

$$F_T = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix}$$

Global Load Vector

$$\Pi = \frac{1}{2}Q^T KQ - Q^T F$$



Element Connectivity Table

Nodes	
1	2
2	3
3	4
4	5

$$\boxed{F_T = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix}}$$

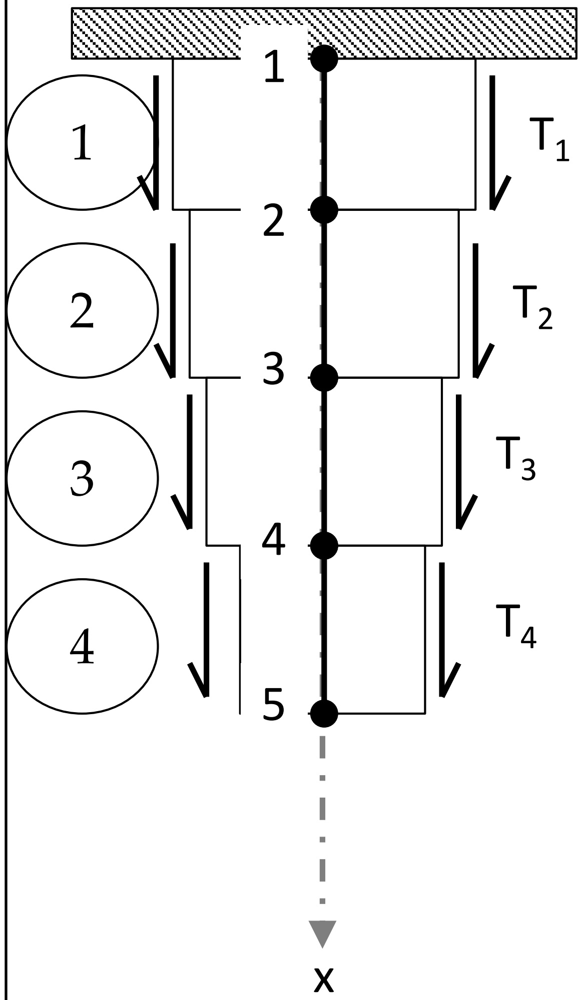
What is the traction force vector for element 1,2,3,4??

$$T^1 = T\frac{l_1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad T^2 = T\frac{l_2}{2}\begin{array}{c} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{array}\begin{array}{c} 2 \\ 3 \end{array}$$

$$T^3 = T\frac{l_3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{array}{c} 3 \\ 4 \end{array}$$

$$T^4 = T\frac{l_4}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{array}{c} 4 \\ 5 \end{array}$$

Global Load Vector



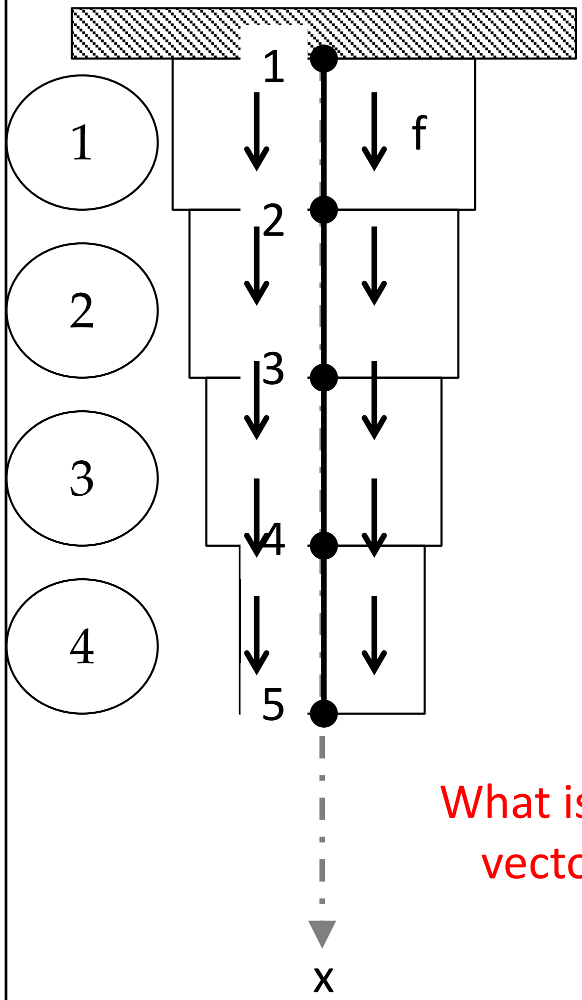
Element Connectivity Table

	Nodes	
	1	2
	2	3
	3	4
	4	5

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F$$

$$F_T = \begin{pmatrix} \dfrac{T_1 l_1}{2} \\ \dfrac{T_1 l_1}{2} + \dfrac{T_2 l_2}{2} \\ \dfrac{T_2 l_2}{2} + \dfrac{T_3 l_3}{2} \\ \dfrac{T_3 l_3}{2} + \dfrac{T_4 l_4}{2} \\ \dfrac{T_4 l_4}{2} \end{pmatrix}$$

Global Load Vector



Element Connectivity Table

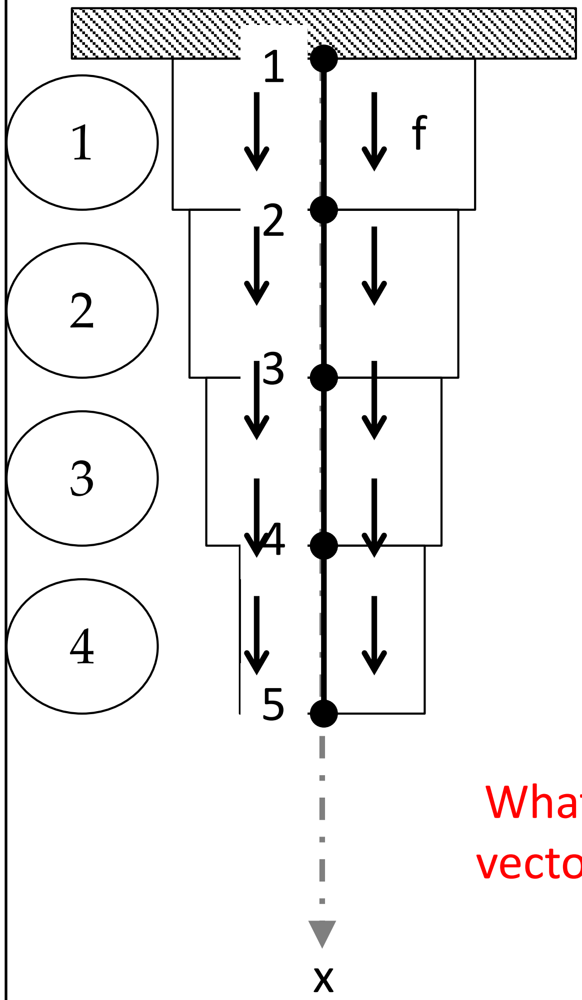
	Nodes	
1	1	2
2	2	3
3	3	4
4	4	5

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F$$

$$F_f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} = \begin{pmatrix} \dfrac{A_1 f_1 l_1}{2} \\ \dfrac{A_1 f_1 l_1}{2} + \dfrac{A_2 f_2 l_2}{2} \\ \dfrac{A_2 f_2 l_2}{2} + \dfrac{A_3 f_3 l_3}{2} \\ \dfrac{A_3 f_3 l_3}{2} + \dfrac{A_4 f_4 l_4}{2} \\ \dfrac{A_5 f_5 l_5}{2} \end{pmatrix}$$

What is the body force vector for element 1,2,3,4??

Global Load Vector



Element Connectivity Table

Nodes	
1	2
2	3
3	4
4	5

What is the point force vector for nodes 1,2,3,4 and 5??

$$\Pi = \frac{1}{2}Q^T K Q - Q^T F$$

$$P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix}$$

Global Load Vector

$$F = F_f + F_T + P = \begin{pmatrix} \dfrac{A_1 f_1 l_1}{2} \\ \dfrac{A_1 f_1 l_1}{2} + \dfrac{A_2 f_2 l_2}{2} \\ \dfrac{A_2 f_2 l_2}{2} + \dfrac{A_3 f_3 l_3}{2} \\ \dfrac{A_3 f_3 l_3}{2} + \dfrac{A_4 f_4 l_4}{2} \\ \dfrac{A_5 f_5 l_5}{2} \end{pmatrix} + \begin{pmatrix} \dfrac{T_1 l_1}{2} \\ \dfrac{T_1 l_1}{2} + \dfrac{T_2 l_2}{2} \\ \dfrac{T_2 l_2}{2} + \dfrac{T_3 l_3}{2} \\ \dfrac{T_3 l_3}{2} + \dfrac{T_4 l_4}{2} \\ \dfrac{T_4 l_4}{2} \end{pmatrix} + \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix} = \begin{pmatrix} \dfrac{A_1 f_1 l_1}{2} + \dfrac{T_1 l_1}{2} \\ \left(\dfrac{A_1 f_1 l_1}{2} + \dfrac{A_2 f_2 l_2}{2}\right) + \left(\dfrac{T_1 l_1}{2} + \dfrac{T_2 l_2}{2}\right) \\ \left(\dfrac{A_2 f_2 l_2}{2} + \dfrac{A_3 f_3 l_3}{2}\right) + \left(\dfrac{T_2 l_2}{2} + \dfrac{T_3 l_3}{2}\right) \\ \left(\dfrac{A_3 f_3 l_3}{2} + \dfrac{A_4 f_4 l_4}{2}\right) + \left(\dfrac{T_3 l_3}{2} + \dfrac{T_4 l_4}{2}\right) \\ \dfrac{A_5 f_5 l_5}{2} + \dfrac{T_4 l_4}{2} \end{pmatrix} + \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix}$$

By the end of the notes:

You are expected to be able:

- To formulate the element stiffness matrix for a 1-D bar element using the potential energy approach