SKMM 3033
Finite Element Method

Topic 3: One-dimensional (1-D) problems

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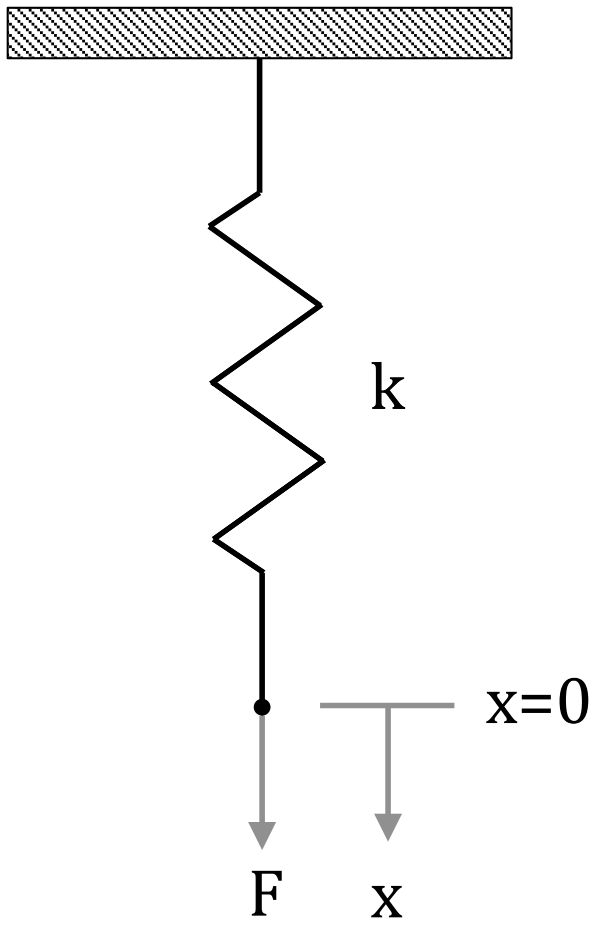
By the end of the notes:

The students are expected:

- To describe shape function for an element displacement based on the natural coordinate

Basic concept of FEM

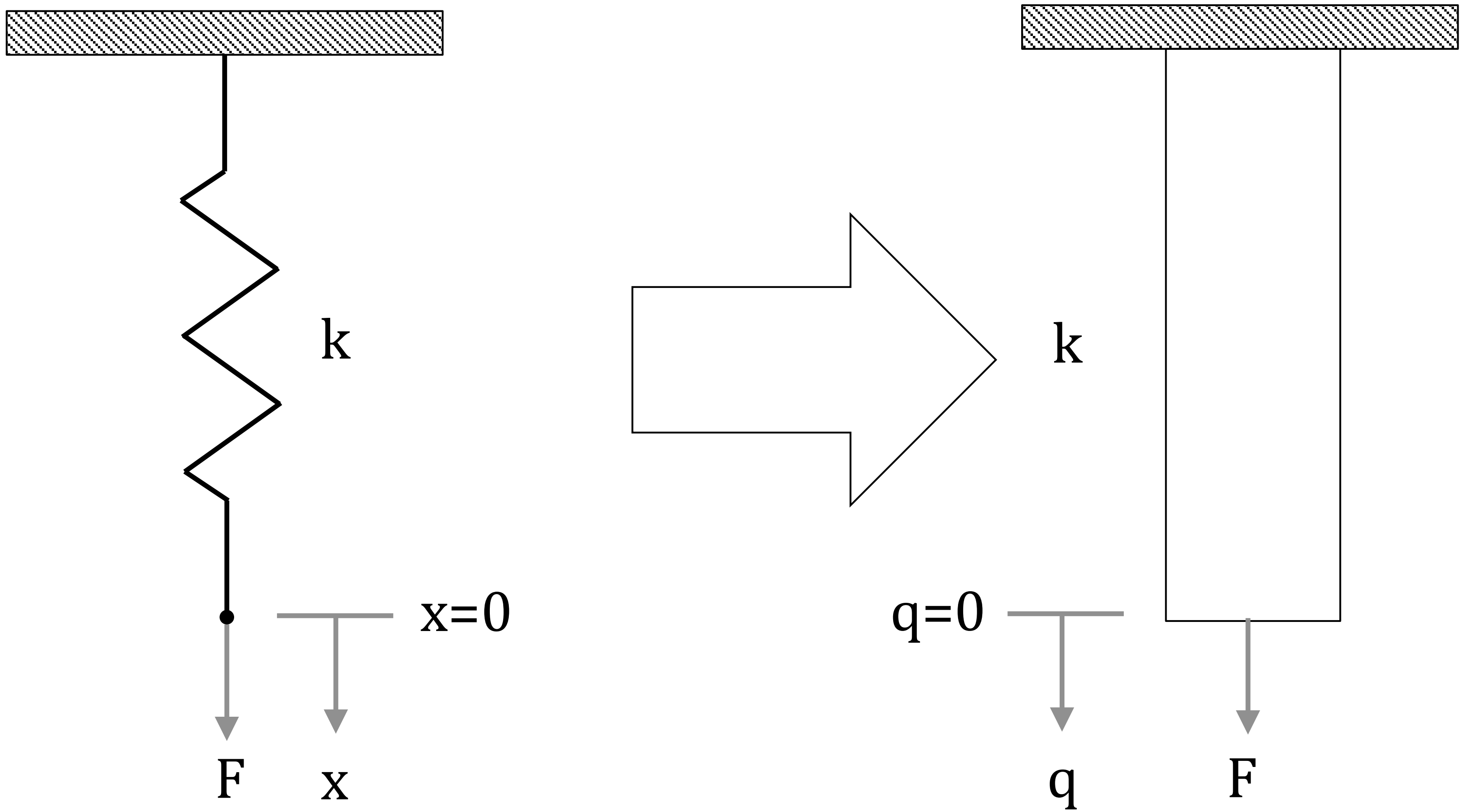


According to Hooke's law

$$F = kx$$

Where k is the spring stiffness

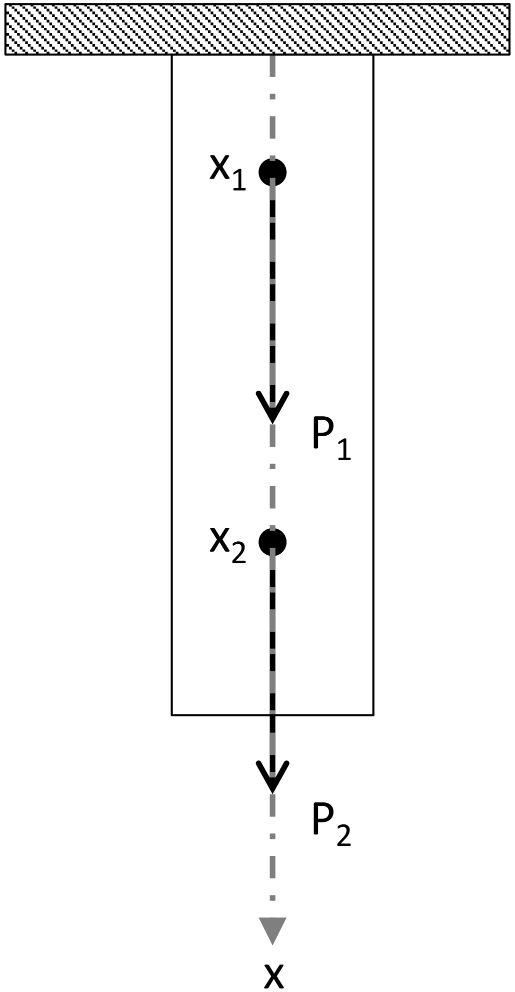
Basic concept of FEM



Loading Condition

Point load, P

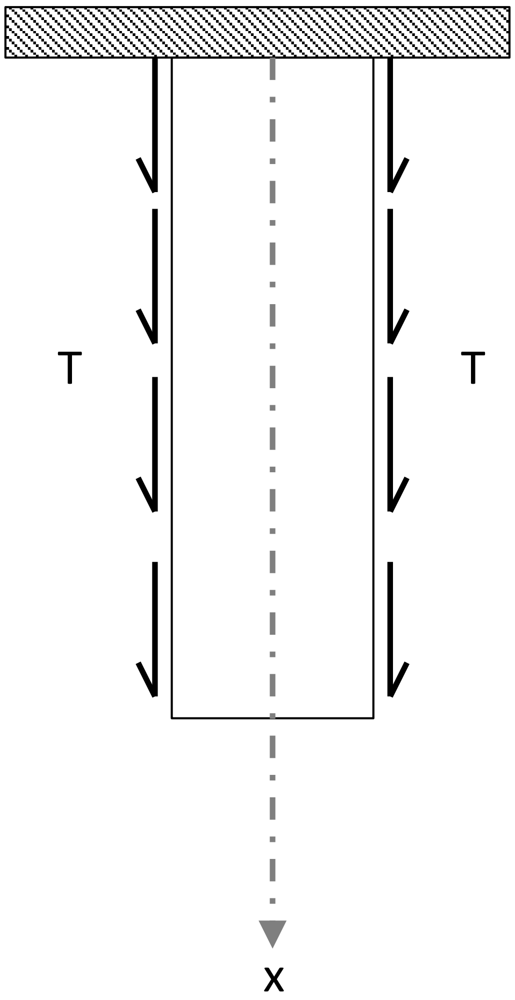
- Concentrated load acting at any point along the body/structure
- Unit in Newton (N)



Loading Condition

Traction force, T

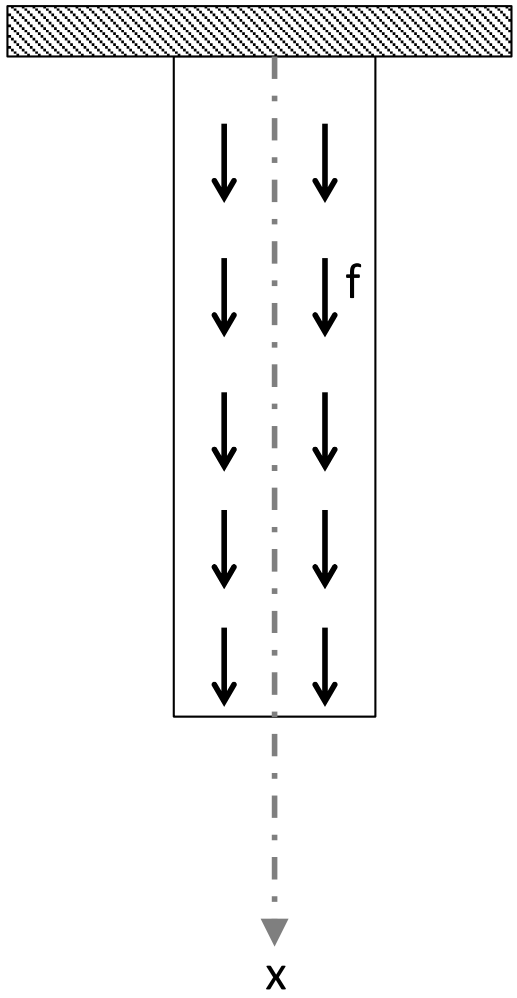
- Distributed load acting on the surface of the body
- For a 1-D problem, it is defined in terms of force per unit length



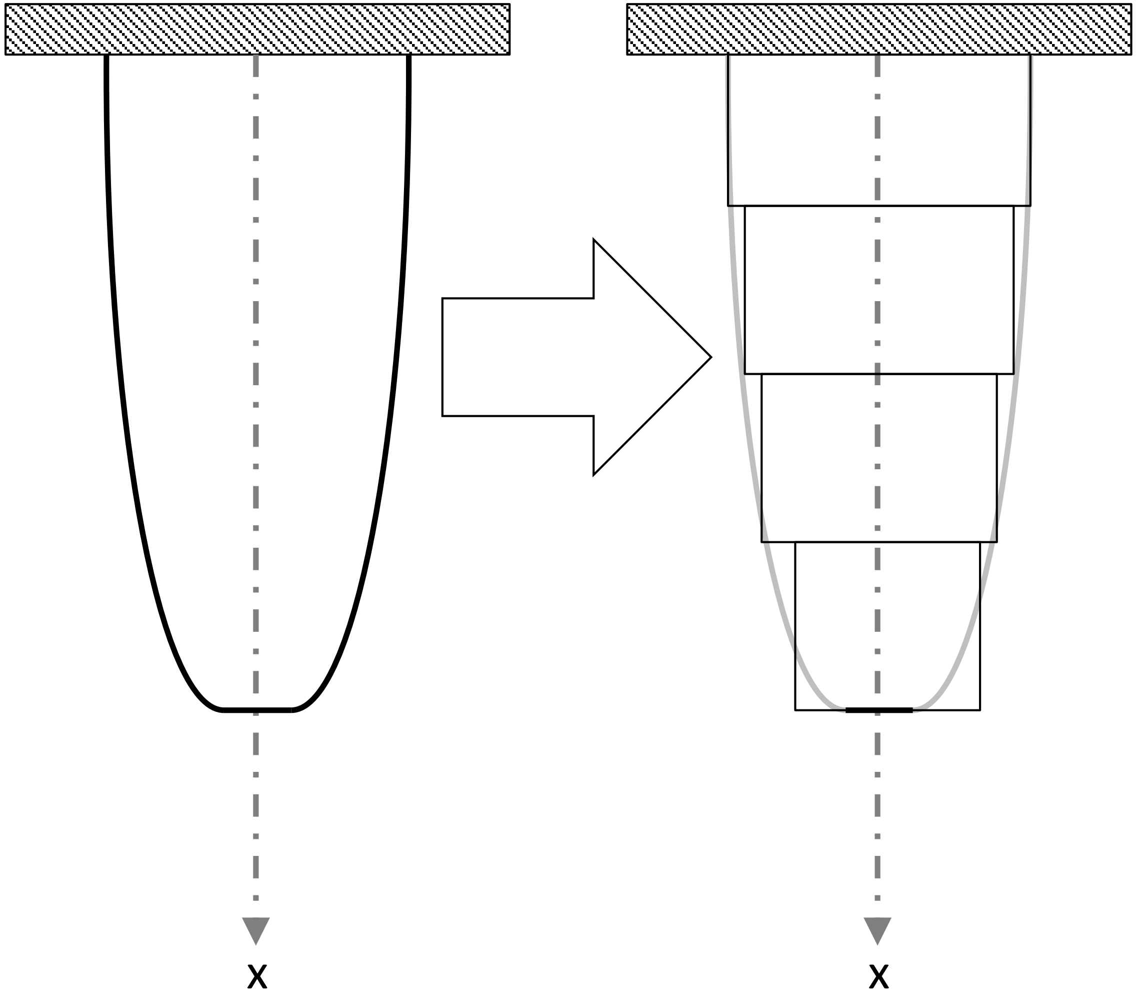
Loading Condition

Body force, f

- Distributed force acting on every elemental volume of the body
- It has the units of force per unit volume
- E.g. self weight of the body due to gravity



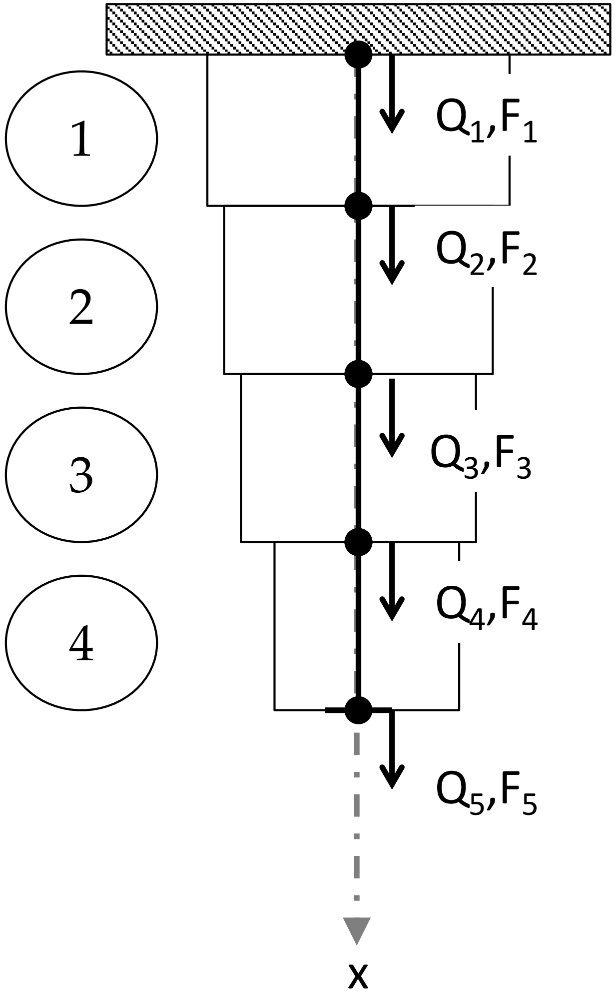
Finite Element Modeling



Step 1:
Element Discretisation

- Subdivide the bar into several sections
- Each section should have uniform cross-sectional area
- The non-uniform bar is then discretised into a stepped bar

Finite Element Modeling



Step 2:
Numbering Scheme

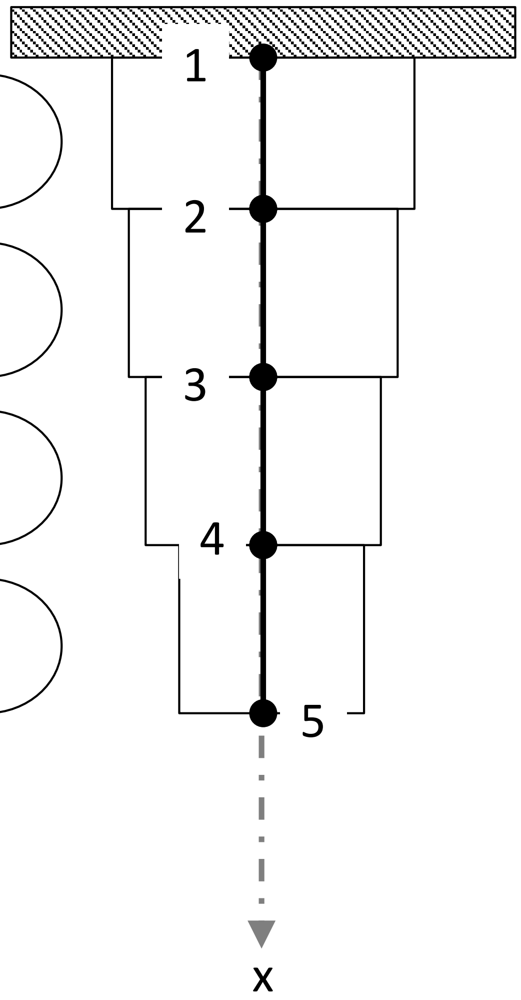
- A global numbering scheme is assigned
- X-direction in this case is considered the ***global coordinate direction***
- F represents the global forces acting at the nodes
- Q represents the global displacements at the nodes

Finite Element Modeling

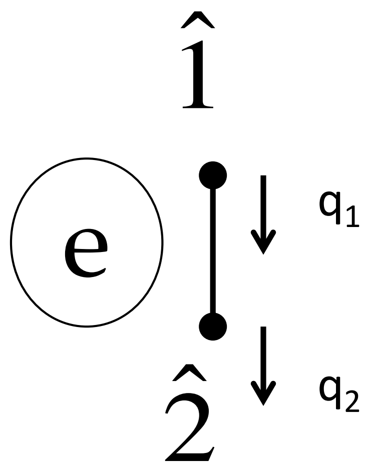
Step 3:
Element Connectivity

Element Connectivity Table

Element	Nodes	
1	1	2
2	2	3
3	3	4
4	4	5



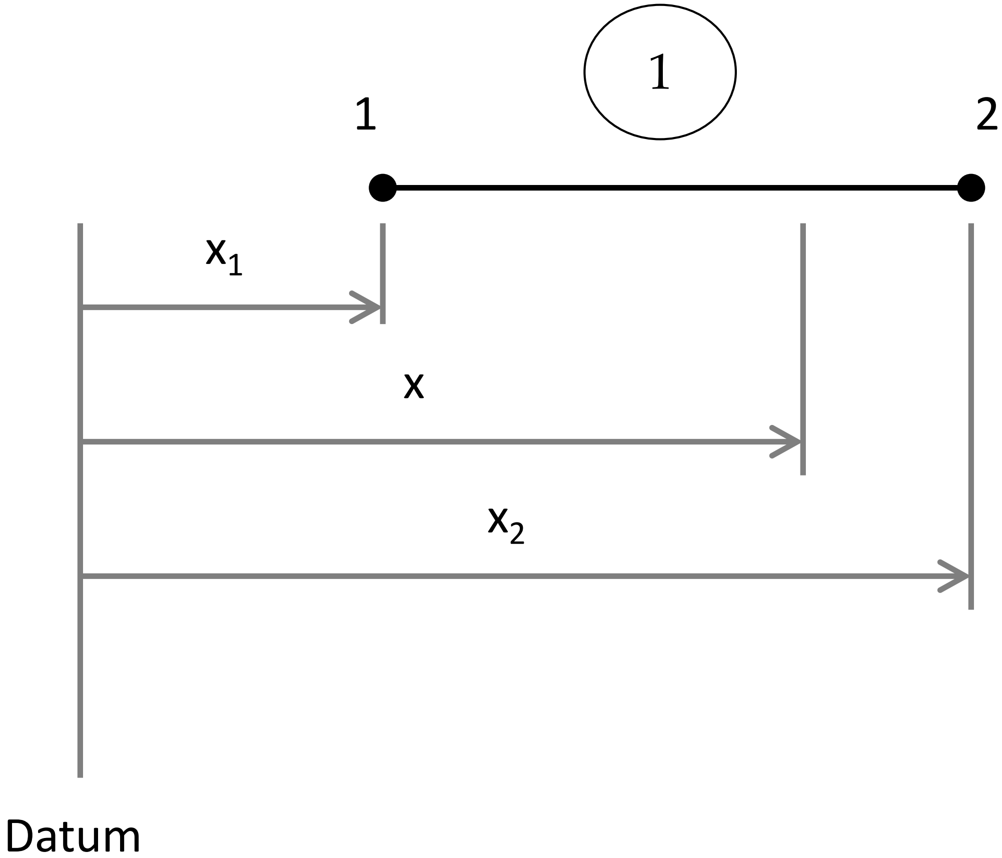
- q represents the nodal displacements in the ***local coordinate direction***



Element Analysis
(1-D element - 2 nodes)

Natural Coordinate

Consider a single element (2 nodes):



Assuming the natural/intrinsic coordinate is ξ:

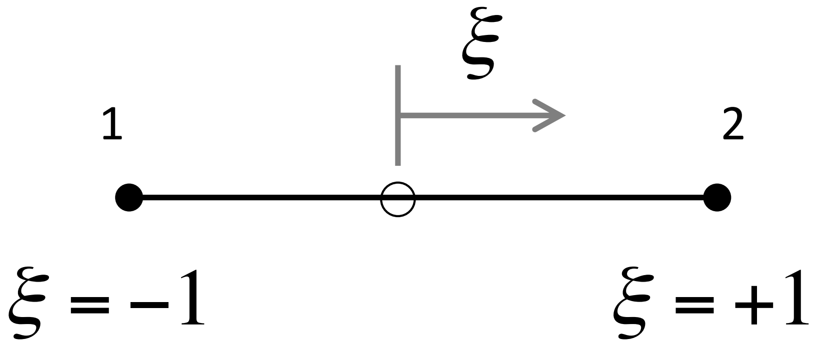
$$\xi = mx + c$$

For a 2-node element, we assume:

$$x_1 \rightarrow \xi = -1$$

$$x_2 \rightarrow \xi = +1$$

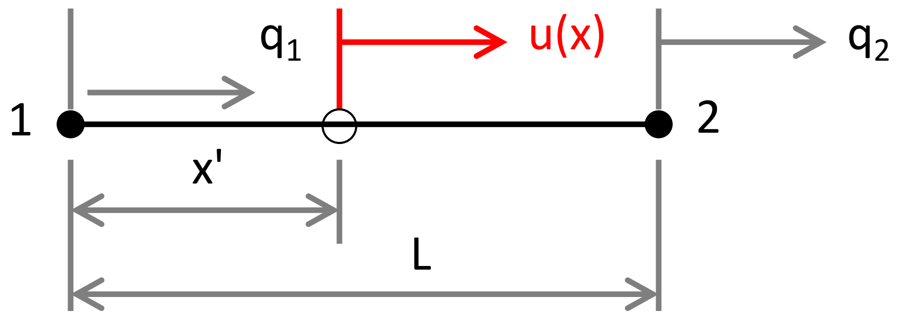
Natural Coordinate



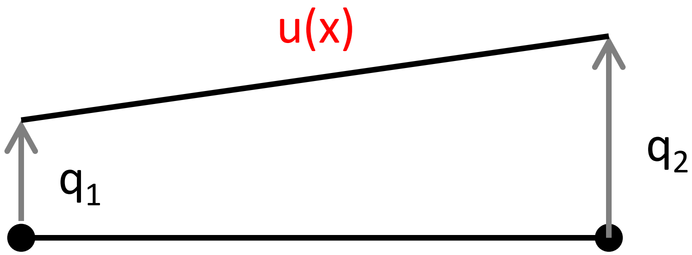
The natural/intrinsic coordinate system can be defined as:

$$\xi = 2\left(\frac{x - x_1}{x_2 - x_1}\right) - 1$$

Element Displacement



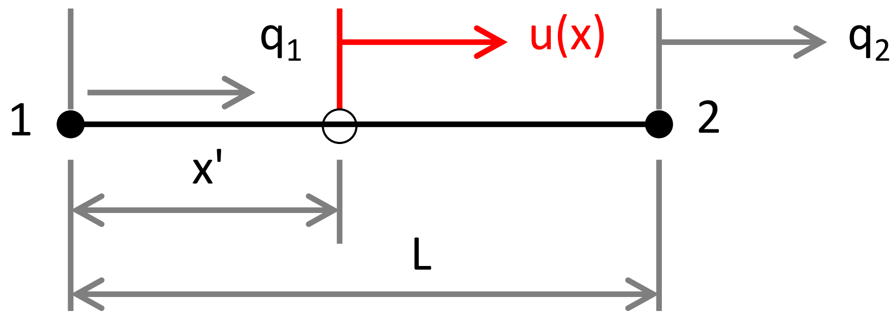
q_1 – displacement at node 1
q_2 – displacement at node 2
u – displacement of a point within the element



- *The displacement field, u(x) within an element is not known*
- *To simplify the problem, it is assumed that the displacement varies linearly from node 1 to node 2*

$$u = a_1 + a_2 x'$$

Element Displacement



q_1 – *displacement at node 1*

q_2 – *displacement at node 2*

u – displacement of a point within the element

$$u = \left(1 - \frac{x'}{L}\right) q_1 + \left(\frac{x'}{L}\right) q_2$$

$$u = \begin{bmatrix} \left(1 - \frac{x'}{L}\right) & \frac{x'}{L} \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

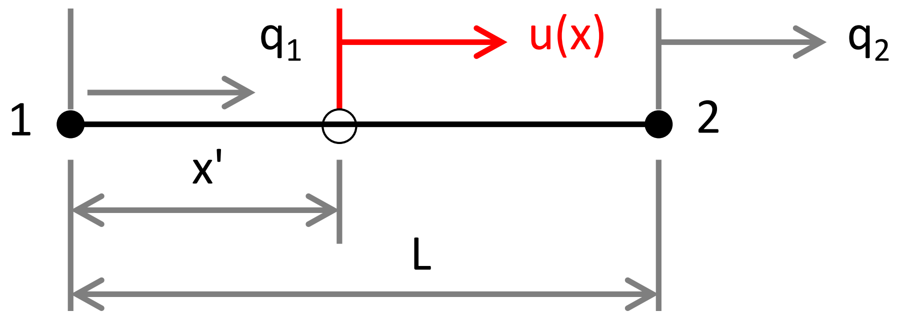
$$\therefore u = \begin{pmatrix} N_1 & N_2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

Shape Functions

Shape functions have to satisfy the following:

- *First derivative MUST be finite within an element*
- *Displacements MUST be continuous across element boundary*

Element Displacement



q_1 – displacement at node 1

q_2 – displacement at node 2

u – displacement of a point within the element

$$\therefore N_1 = 1 - \frac{x'}{L} = 1 - \frac{(\xi+1)}{2} = \frac{(1-\xi)}{2}$$

$$\therefore N_2 = \frac{x'}{L} = \frac{(\xi+1)}{2}$$

To describe the Shape function in terms of Natural Coordinate:

$$\xi = 2\left(\frac{x - x_1}{x_2 - x_1}\right) - 1$$

$$\xi = 2\left(\frac{x'}{L}\right) - 1$$

$$\therefore x' = \frac{L(\xi+1)}{2}$$

Strain-Displacement Relation

Strain-Displacement Relation:

$$\varepsilon = \frac{du}{dx}$$

$$\varepsilon = \frac{1}{x_2 - x_1}\left(-q_1 + q_2\right)$$

$$= \frac{1}{l_e}\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

$$\{B\} = \frac{1}{l_e}\begin{pmatrix} -1 & 1 \end{pmatrix} \rightarrow \varepsilon = \{B\}\{q\}$$

Applying Chain-Rule:

$$\rightarrow \varepsilon = \frac{du}{d\xi} \times \frac{d\xi}{dx}$$

$$\xi = 2\left(\frac{x - x_1}{x_2 - x_1}\right) - 1 \rightarrow \frac{d\xi}{dx} = 2\left(\frac{1}{x_2 - x_1}\right)$$

$$u = \left(\frac{1-\xi}{2}\right)q_1 + \left(\frac{\xi+1}{2}\right)q_2 \rightarrow \frac{du}{d\xi} = \frac{-q_1 + q_2}{2}$$

Stress-Strain Relation

Stress-Strain Relation:

$$\sigma = E\varepsilon$$

$$\therefore \sigma = E\{B\}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = E\{B\}\{q\}$$

Strain-Displacement Relation:

$$\varepsilon = \{B\}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

By the end of the notes:

All of you are expected to be able:

- To describe shape function for an element displacement based on the natural coordinate