SSCE1693 ENGINEERING MATHEMATICS

CHAPTER 4:

IMPROPER INTEGRALS

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4.1 L'Hopital Rule

4.1.1 L'Hopital Rule for 0/0

4.1.2 L'Hopital Rule for ∞/∞

4.2 Improper Integrals

4.2.1 Improper Integral Type 1

4.2.2 Improper Integral Type 2

4.1 L'Hopital Rule

If you are doing any limit and you get something in the form 0/0 or ∞/∞, then you should probably try to use L'Hopital rule. The basic idea of L'Hospital rule is simple.

Consider the limit

$$\lim_{x\to a}\frac{f(x)}{g(x)}.$$

If both the numerator and the denominator are finite at *a* and $g(a)\neq 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f(a)}{g(a)}.$$

Example 4.1:

$$\lim_{x\to 3}\frac{x^2+1}{x+2}=\frac{10}{5}=2.$$

But what happens if both the numerator and the denominator tend to zero?

It is not clear what the limit is. In fact, depending on what functions $f\left(x\right)$ and $g\left(x\right)$ are, the limit can be anything at all!

4.1.1 L'Hopital Rule for 0/0

Suppose $\lim f(x) = \lim g(x) = 0$. Then

1. If $\lim \frac{f'(x)}{g'(x)} = L$, then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)} = L.$$

2. If $\lim \frac{f'(x)}{g'(x)}$ tends to $+\infty$ or $-\infty$ in the limit, so does $\lim \frac{f(x)}{g(x)}$.

Example 4.2:

1. Find $\lim\limits_{x\to 0} \frac{\sin x}{x}$ by L'Hopital rule.

2. Find $\lim\limits_{x\to 1} \frac{2\ln x}{x-1}$.

3. Find $\lim\limits_{x\to 0} \frac{e^x-1}{x}$.

4. Find $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2}$.

Note: If the numerator and the denominator both tend to $+\infty$ or $-\infty$, L'Hopital rule still applies.

4.1.2 L'Hopital Rule for ∞/∞

Suppose $\lim f(x)$ and $\lim g(x)$ are both infinte.

Then

1. If $\lim \dfrac{f'(x)}{g'(x)} = L$, then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)} = L.$$

2. If $\lim \dfrac{f'(x)}{g'(x)}$ tends to $+\infty$ or $-\infty$ in the limit, so does $\lim \dfrac{f(x)}{g(x)}$.

Example 4.3:

1. Find $\lim\limits_{x\to\infty} \dfrac{x}{e^x}$.
2. Find $\lim\limits_{x\to\infty} \dfrac{\ln(\ln x^{1000})}{\ln x}$.

4.2 Improper Integrals

The definite integral

$$\int_a^b f(x)\,dx$$

is known as *improper integral* if either

a) one or both limits are infinite, or

b) $f(x)$ is undefined at certain points on/in the interval.

Note: We called case: a) as Type I

b) as Type II

4.2.1 Improper Integral Type 1

1) If $f(x)$ is continuous in the interval $[a, \infty)$,

then $\int_a^\infty f(x)dx = \lim_{T \to \infty} \int_a^T f(x)\,dx.$

2) If *f(x)* is continuous in the interval $(-\infty, b]$,

then $\int_{-\infty}^b f(x)dx = \lim_{T \to -\infty} \int_T^b f(x)\,dx.$

3) If *f(x)* is continuous in the interval $(-\infty, \infty)$,

then $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$

with any real number *c*.

Note:

- The improper integrals in 1) and 2) is said to *converge* if the limit exists and *diverge* if the limit does not exist.
- The improper integrals in 3) is said to *converge* if both terms converge and *diverge* if either term diverges.

Example 4.4:

1. Determine whether the following integral are convergent or divergent:

   a) $\int_1^\infty \frac{1}{x}\, dx$

   b) $\int_0^\infty xe^{-x}\, dx$

   c) $\int_{-\infty}^\infty \frac{x}{1+x^2}\, dx$

2. For what values of p is the integral $\int_1^\infty \frac{1}{x^p}\, dx$ convergent?

4.2.2 Improper Integral Type 2

1) If $f(x)$ is continuous on $[a, b)$, and discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{T \to b^-} \int_a^T f(x)\, dx \, .$$

2) If $f(x)$ is continuous on $(a, b]$, and discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{T \to a^+} \int_T^b f(x)\, dx.$$

3) If $f(x)$ has discontinuity at c, where $a < c < b$, then

$$\int_a^b f(x)dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

Note:

- The improper integrals in 1) and 2) is said to *converge* if the limit exists and *diverge* if the limit does not exist.
- The improper integrals in 3) is said to *converge* if both terms converge and *diverge* if either term diverges.

Example 4.5:

1. Determine whether $\int_1^9 \frac{1}{\sqrt[3]{x-9}}\, dx$ converge or diverge.
2. Find $\int_0^3 \frac{1}{x-1}\, dx$ if possible.