SSCE1693 ENGINEERING MATHEMATICS

CHAPTER 3:

INTEGRATION

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3.1 Integration of Hyperbolic Functions

3.2 Integration of Inverse Trigonometric Functions

3.3 Integration of Inverse Hyperbolic Functions

***Revision*: Methods involved:**

1. Substitution of *u*
2. By parts
3. Tabular method
4. Partial fractions

REVISION: Techniques of integration

1. Integration by substitution

Example:

a) $\int \frac{\sin x}{1-\cos x}\, dx$

b) $\int \sin x \cos^4 x \; dx$

c) $\int x \cos x^2 \; e^{\sin x^2} \; dx$

2. Integration by parts

Example:

a) $\int x \cos x \, dx$

b) $\int x \sin 2x \, dx$

3. Tabular methods

Example:

a) $\int x \sec^2 x \, dx$

b) $\int e^{3x} \cos 2x \, dx$

4. Integration using partial fractions

Example:

a) $\int \dfrac{3x+2}{x^2+3x+2} dx$

b) $\int \dfrac{1}{x^3+2x^2+x} dx$

3.1 Integrals of Hyperbolic Functions

Integral Formulae

1. $\int \sinh x dx = \cosh x + C$
2. $\int \cosh x dx = \sinh x + C$
3. $\int \operatorname{sech}^2 x dx = \tanh x + C$
4. $\int \operatorname{cosech}^2 x dx = -\coth x + C$
5. $\int \operatorname{sech} x \tanh x dx = -\operatorname{sech} x + C$
6. $\int \operatorname{cosech} x \coth x dx = -\operatorname{cosech} x + C$

Example 3.1 : Evaluate the following integrals

a) $\int \sinh x \cosh x \, dx$

b) $\int \sqrt{\tanh x} \, \text{sech}^2 x \, dx$

c) $\int x \cosh x \, dx$

d) $\int x^3 \cosh x \, dx$

3.2 Integration of Inverse Trigonometric Functions

Integration formulae of the Inverse Trigonometric Functions

Differentiation	Integration
$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$	$\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x+C$
$\frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}}$	$\int\frac{-dx}{\sqrt{1-x^2}}=\cos^{-1}x+C$
$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$	$\int\frac{dx}{1+x^2}=\tan^{-1}x+C$
$\frac{d}{dx}(\cot^{-1}x)=\frac{-1}{1+x^2}$	$\int\frac{-dx}{1+x^2}=\cot^{-1}x+C$
$\frac{d}{dx}(\sec^{-1}x)=\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\int\frac{dx}{\lvert x\rvert\sqrt{x^2-1}}=\sec^{-1}x+C$
$\frac{d}{dx}(\csc^{-1}x)=\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	$\int\frac{-dx}{\lvert x\rvert\sqrt{x^2-1}}=\csc^{-1}x+C$

Example 3.2 : Evaluate the following integrals

a) $\int_0^1 \tan^{-1} x \, dx$

b) $\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}} \, dx$

c) $\int \frac{\sqrt{\tan^{-1} x}}{1+x^2} \, dx$

What about

$$\int \frac{dx}{\sqrt{4-x^2}},\ \int \frac{dx}{9+x^2},\ \int \frac{dx}{|x|\sqrt{x^2-10}}\ \ldots?$$

To find the answer for this question, lets try

to solve $\int \frac{dx}{\sqrt{a^2-x^2}}$.

Solution:

Step 1: Factorize a^2 out of the equation

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{dx}{\sqrt{a^2\left(1-\frac{x^2}{a^2}\right)}}$$

$$= \frac{1}{a}\int \frac{dx}{\sqrt{\left(1-\left(\frac{x}{a}\right)^2\right)}}$$

Step 2: Let $u = {}^{x}/_{a}$, then $du = {}^{1}/_{a}\ dx$

$$= \frac{1}{a}\int \frac{a\,du}{\sqrt{(1-(u)^2)}}$$

$$= \int \frac{du}{\sqrt{(1-u^2)}}$$

Step 3: Integrate the equation and resubstitute $u = {}^{x}/_{a}$

$$= \sin^{-1} u + C$$

$$= \sin^{-1}({}^{x}/_{a}) + C$$

Using the same method, we can find the solution for

$$\int \frac{dx}{a^2 + x^2}, \int \frac{dx}{|x|\sqrt{x^2 - a^2}}, \cdots$$

Example 3.3 : Evaluate the following integrals

1. a) $$\int \frac{dx}{\sqrt{16-x^2}}$$

   b) $$\int \frac{2\,dx}{3+x^2}$$

2. a) $$\int \frac{dx}{\sqrt{1-4x^2}}$$

   b) $$\int \frac{dx}{4+3x^2}$$

3. a) $$\int \frac{dx}{\sqrt{-x^2+2x+3}}$$

   b) $$\int \frac{dx}{x^2-2x+2}$$

3.2 Integration involving Inverse Hyperbolic Functions

Integration formulae of the Inverse Hyperbolic Functions:

Differentiation	Integration
$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$	$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C$
$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$	$\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + C$
$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$	$\int \frac{dx}{1-x^2} = \tanh^{-1} x + C$

What about

$$\int \frac{dx}{\sqrt{4+x^2}}, \int \frac{dx}{x^2-8}, \int \frac{dx}{25-x^2} \ldots?$$

To find the answer for this question, lets try

to solve $\int \frac{dx}{\sqrt{a^2+x^2}}$.

Solution:

Step 1: Factorize a^2 out of the equation

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \int \frac{dx}{\sqrt{a^2\left(1+\frac{x^2}{a^2}\right)}}$$

$$= \frac{1}{a}\int \frac{dx}{\sqrt{\left(1+\left(\frac{x}{a}\right)^2\right)}}$$

Step 2: Let $u = {}^{x}/_{a}$, then $du = {}^{1}/_{a}\ dx$

$$= \frac{1}{a}\int \frac{a\,du}{\sqrt{(1+(u)^2)}}$$

$$= \int \frac{du}{\sqrt{(1+u^2)}}$$

Step 3: Integrate the equation and resubstitute $u = {}^{x}/_{a}$

$$= \sinh^{-1} u + C$$

$$= \sinh^{-1}({}^{x}/_{a}) + C$$

Using the same method, we can find the solution for

$$\int \frac{dx}{a^2 - x^2},\ \int \frac{dx}{\sqrt{x^2 - a^2}},\ \ldots$$

Examples 3.4:

1. Solve the following:

a) $\int \frac{dx}{\sqrt{3x^2+2}}$

b) $\int \frac{dx}{x\sqrt{9-4x^2}}$

c) $\int \frac{dx}{\sqrt{2(x-3)^2+1}}$

d) $\int \frac{dx}{\sqrt{x^2+4x+3}}$

2. Show that $\int \frac{x+1}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1} + \sinh^{-1} x + C$.