Numerical Methods II

SSCM 3423

Chapter 5

This chapter solves boundary value problems (BVP) using finite element method (FEM)

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Finite Element Method (FEM)

Comparison with the finite difference method (FDM)

The finite difference method (FDM) is an alternative way of approximating solutions of PDEs.
The differences between FEM and FDM are:

- The finite difference method is an approximation to the differential equation; the finite element method is an approximation to its solution.
- The most attractive feature of the FEM is its ability to handle complex geometries (and boundaries) with relative ease. While FDM in its basic form is restricted to handle rectangular shapes and simple alterations.
- The most attractive feature of finite differences is that it can be very easy to implement.
- The quality of the approximation between grid points is poor in FDM comparing to FEM.
- The quality of a FEM approximation is often higher than in the corresponding FDM approach, but this is extremely problem dependent and several examples to the contrary can be provided.

Generally, FEM is the method of choice in all types of analysis in structural mechanics while computational fluid dynamics (CFD) tends to use FDM or other methods (e.g., finite volume method). CFD problems usually require discretization of the problem into a large number of cells/gridpoints (millions and more), therefore cost of the solution favors simpler, lower order approximation within each cell.

UTM

Finite Element Method – brief history

- In 1943, Richard Courant introduce an interpolation from continuous system into triangular segments. (The unveiling of ENIAC at the University of Pennsylvania.)
- In the 1950s, a team from Boeing demonstrated that complex surfaces can be analyzed with a matrix of triangular shapes via interpolation.
- Dr. Ray Clough coined the term "finite element" in 1960. In 1960s saw the true beginning of commercial FEA as computers is invented with high computational capability.
- In the early 1960s, the MacNeal-Schwendler Corporation (MSC) develop a general purpose FEA code. This original code had a limit of 68,000 degrees of freedom. When the NASA contract was completed, MSC continued development of its own version called MSC/NASTRAN, while the original NASTRAN become available to the public and formed the basis of the FEA packages available today. Around the time MSC/NASTRAN was released, ANSYS, MARC, and SAP were introduced.

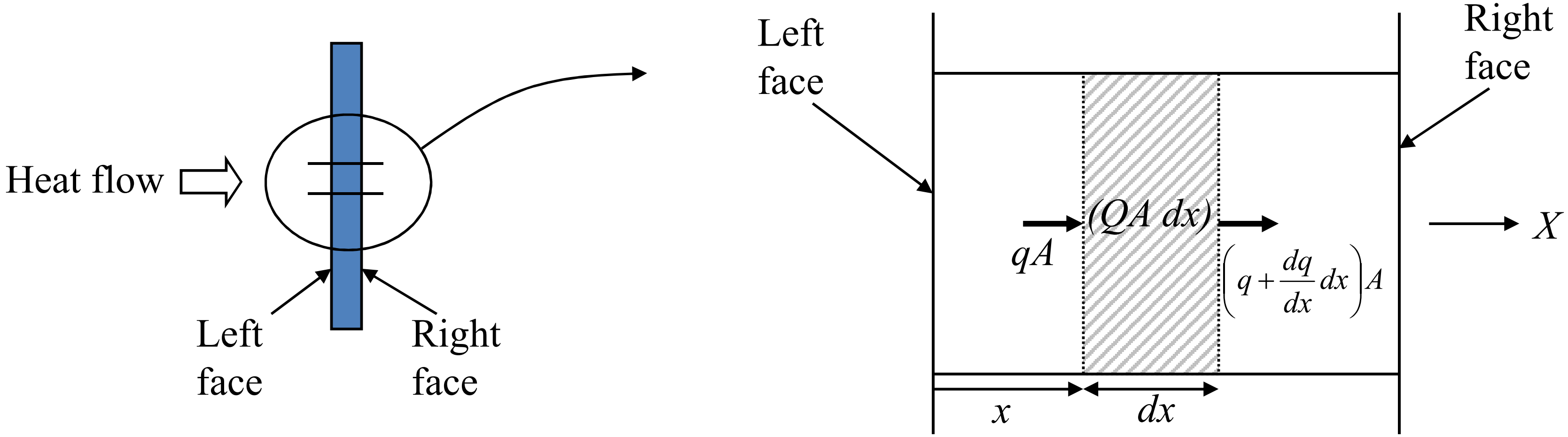
Finite Element Method – brief history

- In the 1970s, Computer-aided design (CAD) was introduced.
- In the 1980s, the use of FEA and CAD on the same workstation with developing geometry standards such as IGES and DXF (type of files). Permitted limited geometry transfer between the systems or programs.
- In the 1980s,CAD progressed from a 2D drafting tool to a 3D surfacing tool, and then to a 3D solid modeling system. Design engineers began to seriously consider in incorporating FEA into the general product design process.
- In the 1990s, the PC platform has become a major force in high- end analysis. The technology has become to accessible that it is actually being “hidden” inside CAD packages. (background calculation)

Finite Element Method

Steady-state 1-D heat conduction

Governing equation (heat conduction in plane wall with uniform heat generation)



Let A =area normal to direction of heat flow,

Q(W/m^3) =internal heat generated per unit volume.

Heat rate (heat flux × area) enter the control volume + heat rate generated = Heat rate leaving control volume.

$$qA + QAdx = \left(q + \frac{dq}{dx}dx\right)A \xrightarrow{\text{simplify}} Q = \frac{dq}{dx}$$

$$q = -k\frac{small - big}{dx} = +ve$$

+ve=heat flux same direction with x-axis

Substitute Fourier's law $q = -k\dfrac{dT}{dx}$ $\Rightarrow$ $\dfrac{d}{dx}\left(k\dfrac{dT}{dx}\right) + Q = 0$

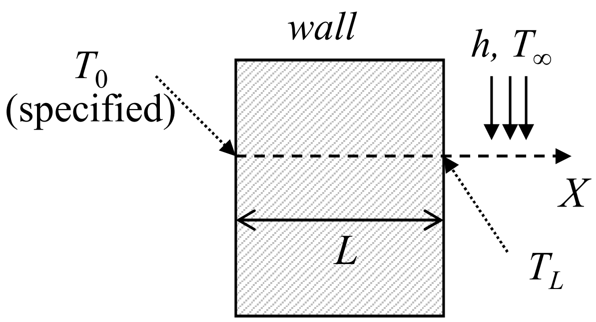
Q is called source when +ve (heat is generated) and is called sink when –ve (heat is consumed)

Here, Q is referred as source.

Finite Element Method

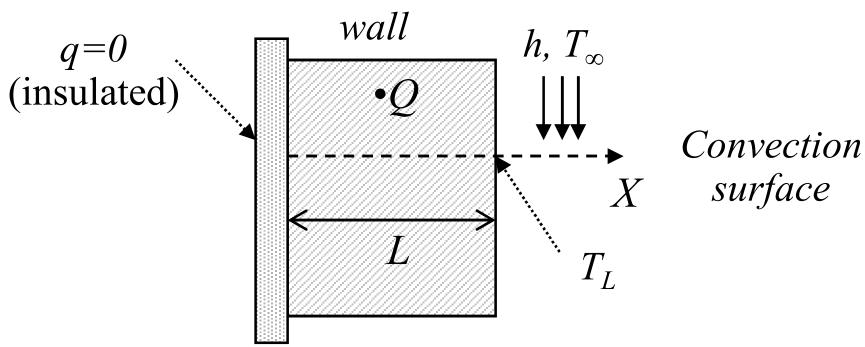
Steady-state 1-D heat conduction, Boundary conditions

Specified temperature



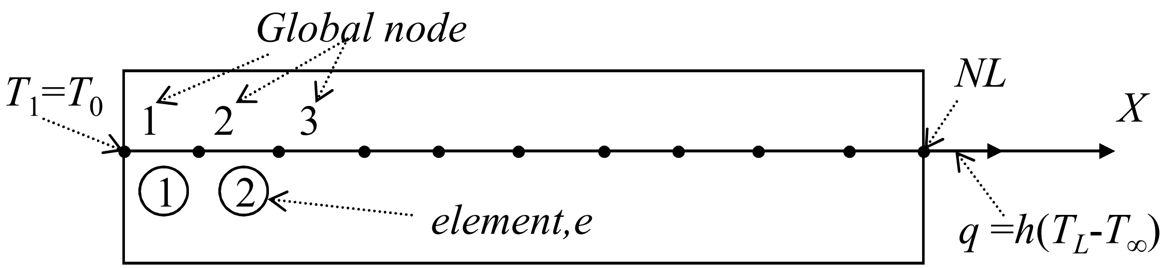
Wall of tank contain hot liquid at T_0,
airstream of T_∞ passed on outside,
maintain T_L at boundary.
$T|_{x=0} = T_0$, $q|_{x=L} = h(T_L - T_\infty)$. [note: $T_L > T_\infty$]

Specified heat flux



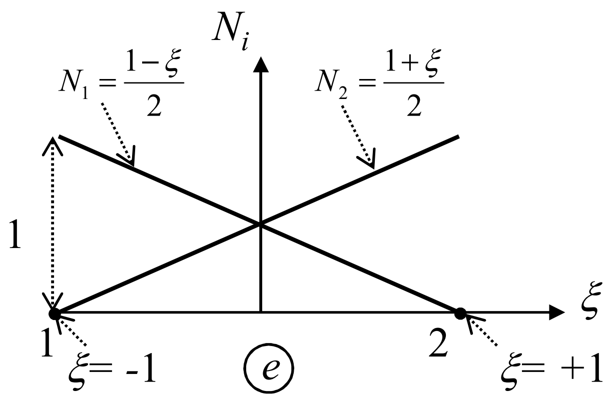
A wall where the inside surface is insulated
And outside is convection surface.
$q|_{x=0} = 0$, $q|_{x=L} = h(T_L - T_\infty)$.

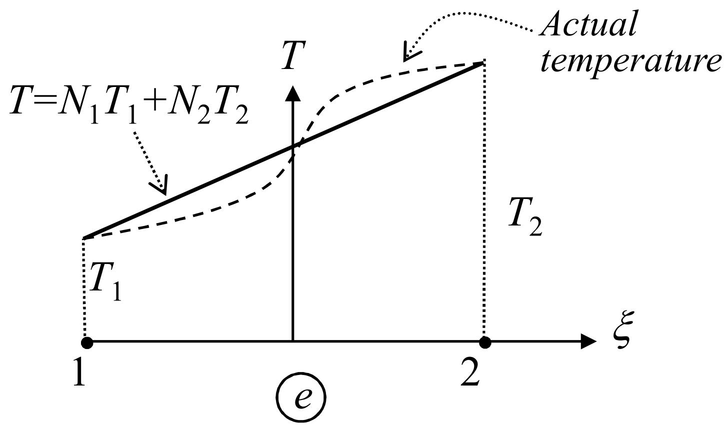
1-D element : two-node element with linear shape functions



Finite Element Method

1-D element





$$T(\xi)=N_1T_1+N_2T_2 = \mathbf{N}\mathbf{T}^e$$

where N_1=(1-ξ)/2, N_2=(1+ ξ)/2, ξ varies from -1 to +1, N=[N_1, N_2], $\mathbf{T}^e$=[T_1, T_2]$^\mathrm{T}$.

Please note $\xi = \frac{2}{x_2 - x_1}(x - x_1) - 1, \; d\xi = \frac{2}{x_2 - x_1}dx = \frac{2}{l_e}dx.$

$$x=N_1x_1+N_2x_2$$

$$x = \frac{(1-\xi)}{2}x_1 + \frac{(1+\xi)}{2}x_2$$

Use chain rule, $\frac{dT}{dx} = \frac{dT}{d\xi}\cdot\frac{d\xi}{dx} = \frac{2}{x_2 - x_1}\frac{d\mathbf{N}}{d\xi}\cdot\mathbf{T}^e = \frac{1}{x_2 - x_1}[-1,1]\mathbf{T}^e = \mathbf{B}_T\mathbf{T}^e.$

where $\mathbf{B}_T = \frac{d}{dx}\mathbf{N} = \frac{1}{x_2 - x_1}[-1,\, 1] = \frac{1}{l_e}[-1 \quad 1]$

$$\int_e f dx = \int_{-1}^{1} f\, J d\xi \;, \quad J = \frac{l_e}{2} = Jacobian$$

Finite Element Method

Galerkin's approach for heat conduction

$y=T=$temperature

Problem: $\frac{d}{dx}\left(k\frac{dy}{dx}\right)+Q=0 \qquad y|_{x=0}=y_0,\quad q|_{x=L}=h(y_L\text{-}y_\infty).$

Assume: $\int_0^L \phi\left[\frac{d}{dx}\left(k\frac{dy}{dx}\right)+Q\right]dx=0$

$\phi(x)$ constructed from same basis function of y, with $\phi(0)=0$. ϕ as a virtual temperature change that is consistent with boundary conditions.

Weighted-Residual Method

First term use integration by part: $\int_{x=a}^{x=b} u\,dv = uv\Big|_{x=a}^{x=b} - \int_{x=a}^{x=b} v\,du \Rightarrow$

$$\phi k\frac{dy}{dx}\Bigg|_{x=0}^{x=L} - \int_0^L k\frac{d\phi}{dx}\frac{dy}{dx}dx + \int_0^L \phi Q\,dx = 0$$

Now, $\phi k\frac{dy}{dx}\Big|_0^L = \phi(L)k(L)\frac{dy}{dx}(L) - \phi(0)k(0)\frac{dy}{dx}(0)$

Since, $q=-k\frac{dy}{dx}$ So, $\phi(0)=0$, $q(L)=-k(L)[dy(L)/dx]=h(y_L\text{-}y_\infty)$, we get $\phi k\frac{dy}{dx}\Big|_0^L = -\phi(L)h(y_L-y_\infty)$.

Finally, we get

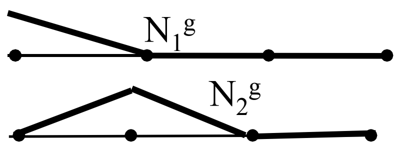
$$-\phi(L)h(y_L-y_\infty) - \int_0^L k\frac{d\phi}{dx}\frac{dy}{dx}dx + \int_0^L \phi Q\,dx = 0$$

⇐ Weak form – reduced (weakened) continuity of y

A global virtual-temperature vector is denoted: $\psi=[\psi_1, \psi_2, \ldots, \psi_{NL}]^T$, or element-wise: $\psi^e=[\psi_i, \psi_{i+1}]^T$.

The test function within each element is interpolated as: (global nodes) $\phi=N\psi$, or element-wise $\phi^e=N^e\psi^e$.

$$\frac{d\phi^e}{dx}=\frac{d}{dx}\phi^e=\frac{d}{dx}\left(\mathbf{N}^e\boldsymbol{\psi}^e\right)=\left(\frac{d\mathbf{N}^e}{d\xi}\frac{d\xi}{dx}\right)\cdot\boldsymbol{\psi}^e=\mathbf{B}_T\boldsymbol{\psi}^e$$



Finite Element Method

Galerkin's approach for heat conduction

<u>Some matrix concept</u>: $(AB)^T = B^T A^T$.
Let A,B,C,D=row vector, AB^T=scalar $\rightarrow AB^T = (AB^T)^T$.
$AB^TCD^T = (AB^T)^T CD^T = B^{TT}A^TCD^T = B(A^TC)D^T$=scalar

$N_i(x_j) = \delta_{ij}$ (Kronecker delta function,global)

We get,

$$-\phi(L)h(y_L - y_\infty) - \int_0^L k \frac{d\phi}{dx}\frac{dy}{dx}dx + \int_0^L \phi Q dx = 0 = -(\mathbf{N}(L)\boldsymbol{\psi})h(y_L - y_\infty) - \sum_{i=1}^{NL-1}\int_{x_i}^{x_{i+1}} k\frac{d\phi}{dx}\frac{dy}{dx}dx + \sum_{i=1}^{NL-1}\int_{x_i}^{x_{i+1}} \phi Q dx$$

$$= -\psi_{NL}h(y_L - y_\infty) - \sum_e \frac{k_e l_e}{2}\int_{-1}^{1}\frac{d\phi^e}{dx}\frac{dy^e}{dx}d\xi + \sum_e \frac{Q_e l_e}{2}\int_{-1}^{1}\mathbf{N}^e\boldsymbol{\psi}^e d\xi = 0$$

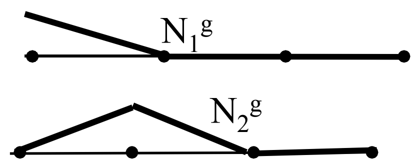
$$d\xi = \frac{2}{l_e}dx$$

Note that, $\frac{d\phi^e}{dx}\frac{dy^e}{dx} = (\mathbf{B}_T\boldsymbol{\psi}^e)(\mathbf{B}_T\mathbf{y}^e) = (\mathbf{B}_T\boldsymbol{\psi}^e)^T(\mathbf{B}_T\mathbf{y}^e) = \boldsymbol{\psi}^T(\mathbf{B}_T^T\mathbf{B}_T)\mathbf{y}^e$ and, $N^e\psi^e$=scalar=$(N^e\psi^e)^T = \psi^T N^T$.

$$0 = -\psi_{NL}h(y_L - y_\infty) - \sum_e \frac{k_e l_e}{2}\int_{-1}^{1}\boldsymbol{\psi}^T(\mathbf{B}_T^T\mathbf{B}_T)\mathbf{y}^e d\xi + \sum_e \frac{Q_e l_e}{2}\int_{-1}^{1}\boldsymbol{\psi}^T\mathbf{N}^T d\xi$$

$$0 = -\psi_{NL}h(y_L - y_\infty) - \sum_e \boldsymbol{\psi}^T\frac{k_e l_e}{2}\int_{-1}^{1}\mathbf{B}_T^T\mathbf{B}_T d\xi\,\mathbf{y}^e + \sum_e \boldsymbol{\psi}^T\frac{Q_e l_e}{2}\int_{-1}^{1}\mathbf{N}^T d\xi$$

$$\mathbf{k}_T = k_e\int \mathbf{B}_T^T\mathbf{B}_T dx = k_e\int \frac{d}{dx}(\mathbf{N}^T)\frac{d}{dx}(\mathbf{N})dx$$



Note that: $\int_{-1}^{1}\mathbf{B}_T^T\mathbf{B}_T d\xi = \frac{1}{l_e^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\int_{-1}^{1}d\xi = \frac{2}{l_e^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$, $\int_{-1}^{1}\mathbf{N}^T d\xi = \int_{-1}^{1}\begin{Bmatrix}(1-\xi)/2\\ (1+\xi)/2\end{Bmatrix}d\xi = \begin{bmatrix}1\\ 1\end{bmatrix}$.

Finally, $$0 = -\psi_{NL}h(y_L - y_\infty) - \sum_e \boldsymbol{\psi}^T\mathbf{k}_T\mathbf{y}^e + \sum_e \boldsymbol{\psi}^T\mathbf{r}_Q$$ where, $\mathbf{k}_T = \frac{k_e}{l_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$, $\mathbf{r}_Q = \frac{Q_e l_e}{2}\begin{bmatrix}1\\ 1\end{bmatrix}$.

Finite Element Method

Galerkin's approach for heat conduction

Some matrix concept: AMC+ANC=(AM+AN)C=A(M+N)C.

Let: $\mathbf{k}_T^{e=i}=\begin{bmatrix} k_{i,i} & k_{i,i+1} \\ k_{i+1,i} & k_{i+1,i+1} \end{bmatrix},\quad \mathbf{r}_Q^{e=i}=\begin{bmatrix} r_i \\ r_{i+1} \end{bmatrix}.$

Please note: k_T are symmetry! $\rightarrow k_{i,j}=k_{j,i}$

2 elements example:

$$\boldsymbol{\psi}^T\mathbf{k}_T^{e=1}\mathbf{y}^{e=1}+\boldsymbol{\psi}^T\mathbf{k}_T^{e=2}\mathbf{y}^{e=2}=\begin{bmatrix}\psi_1 & \psi_2\end{bmatrix}\begin{bmatrix} k_{1,1} & k_{1,2} \\ k_{2,1} & k_{2,2} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}+\begin{bmatrix}\psi_2 & \psi_3\end{bmatrix}\begin{bmatrix} k_{2,2} & k_{2,3} \\ k_{3,2} & k_{3,3} \end{bmatrix}\begin{bmatrix} y_2 \\ y_3 \end{bmatrix}$$

$$=\begin{bmatrix}\psi_1 & \psi_2\end{bmatrix}\begin{bmatrix} k_{1,1} & k_{1,2} & 0 \\ k_{2,1} & k_{2,2} & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}+\begin{bmatrix}\psi_2 & \psi_3\end{bmatrix}\begin{bmatrix} 0 & k_{2,2} & k_{2,3} \\ 0 & k_{3,2} & k_{3,3} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$=\begin{bmatrix}\psi_1 & \psi_2 & \psi_3\end{bmatrix}\begin{bmatrix} k_{1,1} & k_{1,2} & 0 \\ k_{2,1} & k_{2,2} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}+\begin{bmatrix}\psi_1 & \psi_2 & \psi_3\end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_{2,2} & k_{2,3} \\ 0 & k_{3,2} & k_{3,3} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$=\begin{bmatrix}\psi_1 & \psi_2 & \psi_3\end{bmatrix}\begin{bmatrix} k_{1,1} & k_{1,2} & 0 \\ k_{2,1} & 2k_{2,2} & k_{2,3} \\ 0 & k_{3,2} & k_{3,3} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\boldsymbol{\Psi}^T\mathbf{K}_T\mathbf{y}.$$

We also get:

$$\boldsymbol{\psi}^T\mathbf{r}_Q^{e=1}+\boldsymbol{\psi}^T\mathbf{r}_Q^{e=2}=\begin{bmatrix}\psi_1 & \psi_2\end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}+\begin{bmatrix}\psi_2 & \psi_3\end{bmatrix}\begin{bmatrix} r_2 \\ r_3 \end{bmatrix}=\begin{bmatrix}\psi_1 & \psi_2 & \psi_3\end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ 0 \end{bmatrix}+\begin{bmatrix}\psi_1 & \psi_2 & \psi_3\end{bmatrix}\begin{bmatrix} 0 \\ r_2 \\ r_3 \end{bmatrix}$$

$$=\begin{bmatrix}\psi_1 & \psi_2 & \psi_3\end{bmatrix}\begin{bmatrix} r_1 \\ 2r_2 \\ r_3 \end{bmatrix}=\boldsymbol{\Psi}^T\mathbf{R}=\begin{bmatrix}\psi_1 & \psi_2 & \psi_3\end{bmatrix}\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}.$$

Finite Element Method

Galerkin's approach for heat conduction

Finally, we get:

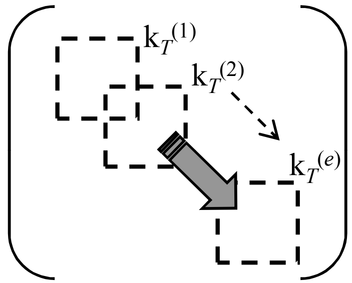
$$0 = -\psi_{NL} h(y_L - y_\infty) - \sum_e \boldsymbol{\psi}^T \mathbf{k}_T \mathbf{y}^e + \sum_e \boldsymbol{\psi}^T \mathbf{r}_Q$$

$$0 = -\psi_{NL} h y_L + \psi_{NL} h y_\infty - \left(\boldsymbol{\psi}_{e=1}^T \mathbf{k}_T^{e=1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \cdots + \boldsymbol{\psi}_{e=NL-1}^T \mathbf{k}_T^{e=NL-1} \begin{bmatrix} y_{NL-1} \\ y_{NL} \end{bmatrix} \right)$$

$$+ \left(\begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} + \cdots + \begin{bmatrix} \psi_{NL-1} & \psi_{NL} \end{bmatrix} \begin{bmatrix} r_{NL-1} \\ r_{NL} \end{bmatrix} \right)$$

$$0 = -\psi_{NL} h y_L + \psi_{NL} h y_\infty - \boldsymbol{\Psi}^T \mathbf{K}_T \mathbf{y} + \boldsymbol{\Psi}^T \mathbf{R}.$$

$K_T =$



The global matrices K_T and R are assembled from element matrices k_T and r_Q.

Now, let $\Psi^T = [\psi_1, \psi_2, \ldots, \psi_{NL}] = [0,1,0,\ldots,0]$, and $y_1 = y_0$, we get

$$-0 + 0 - \begin{bmatrix} K_{21} & K_{22} & \cdots & K_{2,NL} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{NL} \end{bmatrix} + R_2 = 0 \rightarrow \begin{bmatrix} K_{22} & K_{23} & \cdots & K_{2,NL} \end{bmatrix} \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ y_{NL} \end{bmatrix} = R_2 - K_{21} y_0.$$

Continue the process, finally let $\Psi^T = [\psi_1, \psi_2, \ldots, \psi_{NL}] = [0,0,\ldots,1]$, we get ($y_L = y_{NL}$)

$$-1 \cdot h y_L + 1 \cdot h y_\infty - \begin{bmatrix} K_{NL,1} & K_{NL,2} & \cdots & K_{NL,NL} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{NL} \end{bmatrix} + R_{NL} = 0 \rightarrow \begin{bmatrix} K_{NL,2} & K_{NL,3} & \cdots & (K_{NL,NL} + h) \end{bmatrix} \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ y_{NL} \end{bmatrix} = (R_{NL} + h y_\infty) - K_{NL,1} y_0.$$

Finite Element Method

Galerkin's approach for heat conduction

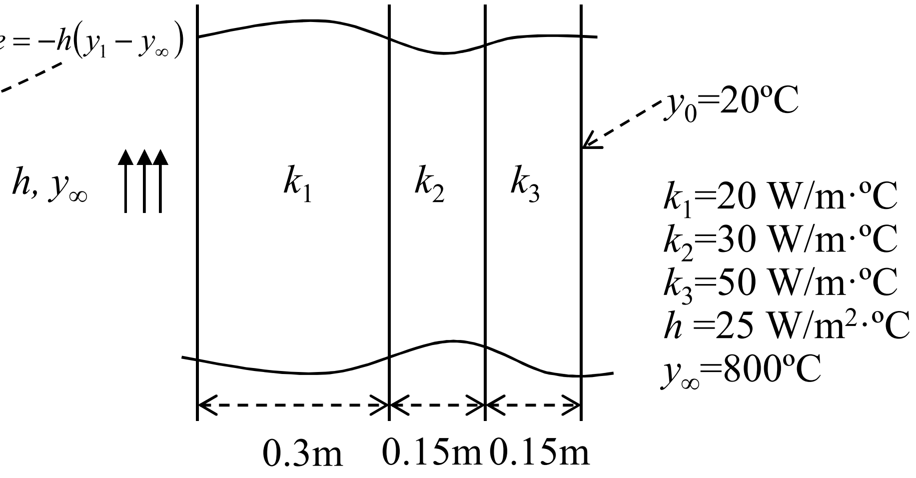
Finally, the compact form is given:

$$\begin{bmatrix} K_{2,2} & K_{2,3} & \cdots & K_{2,NL} \\ K_{3,2} & K_{3,3} & \cdots & K_{3,NL} \\ \vdots & \vdots & \ddots & \vdots \\ K_{NL,2} & K_{NL,3} & \cdots & (K_{NL,NL}+h) \end{bmatrix} \begin{Bmatrix} y_2 \\ y_3 \\ \vdots \\ y_{NL} \end{Bmatrix} = \begin{Bmatrix} R_2 \\ R_3 \\ \vdots \\ R_{NL}+hy_\infty \end{Bmatrix} - \begin{Bmatrix} K_{2,1}y_0 \\ K_{3,1}y_0 \\ \vdots \\ K_{NL,1}y_0 \end{Bmatrix}.$$

Try insulation at $x=L$, $\phi(L)=0$
Try $Q=2$

Problem: A composite wall consists of 3 materials. The outer temperature is y_0=20ºC. Convection heat transfer takes place on the inner surface of the wall with y_∞=800ºC and h=25 W/m²·ºC. Determine the temperature distribution in the wall.

$$q(0) \approx -k\left.\frac{\partial y}{\partial x}\right|_{x_1} = -k\frac{y_{1.5}-y_1}{\Delta x} \approx -k\frac{400-500}{+0.1} = +ve = -h(y_1 - y_\infty)$$



Solution: we use 3 elements of linear element.

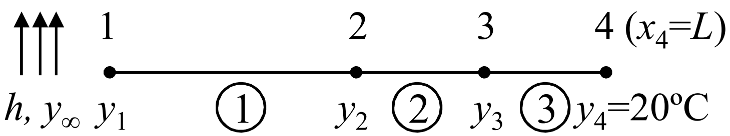
B.C.: $y_4 = y_0$=20, $q|_{x=0} = -h(y_1 - y_\infty)$. [$y_\infty > y_1$] We get

$$\int_0^L \phi\left[\frac{d}{dx}\left(k\frac{dy}{dx}\right)+Q\right]dx = 0 \rightarrow \left.\phi k\frac{dy}{dx}\right|_{x=0}^{x=L} - \int_0^L k\frac{d\phi}{dx}\frac{dy}{dx}dx + \int_0^L \phi Q dx = 0$$

$$\left.\phi k\frac{dy}{dx}\right|_0^L = \phi(L)k(L)\frac{dy}{dx}(L) - \phi(0)k(0)\frac{dy}{dx}(0)$$

So, let $\phi(L)=0$, $q(0) = -k(0)[dy(0)/dx] = -h(y_1 - y_\infty)$, we get

$$\left.\phi k\frac{dy}{dx}\right|_0^L = -\phi(0)h(y_1 - y_\infty).$$



3 elements of linear FE

Finite Element Method

Galerkin's approach for heat conduction

Let ϕ=NΨ, we get

$$0=-\psi_1 h(y_1-y_\infty)-\sum_e \frac{k_e l_e}{2}\int_{-1}^{1}\frac{d\phi^e}{dx}\frac{dy^e}{dx}d\xi+\sum_e \frac{Q_e l_e}{2}\int_{-1}^{1}\mathbf{N}^e\boldsymbol{\psi}^e d\xi$$

Finally,

$$\boxed{0=-\psi_1 h(y_1-y_\infty)-\sum_e \boldsymbol{\psi}^T\mathbf{k}_T\mathbf{y}^e+\sum_e \boldsymbol{\psi}^T\mathbf{r}_Q} \Rightarrow \boxed{0=-\psi_1 h y_1+\psi_1 h y_\infty-\boldsymbol{\Psi}^T\mathbf{K}_T\mathbf{y}+\boldsymbol{\Psi}^T\mathbf{R}.}$$

Now, let $\Psi^T=[\psi_1, \psi_2, \ldots, \psi_{NL}]=[1,0,0,0]$, and $y_4=y_0$, we get

$$-h(y_1-y_\infty)-\begin{bmatrix}K_{11} & K_{12} & K_{13} & K_{14}\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}+R_1=0\rightarrow\begin{bmatrix}(K_{11}+h) & K_{12} & K_{13}\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=R_1+hy_\infty-K_{14}y_0$$

let $\Psi^T=[\psi_1, \psi_2, \ldots, \psi_{NL}]=[0,1,0,0]$, we get

$$-0+0-\begin{bmatrix}K_{2,1} & K_{2,2} & K_{2,3} & K_{2,4}\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}+R_2=0\rightarrow\begin{bmatrix}K_{2,1} & K_{2,2} & K_{2,3}\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=R_2-K_{2,4}y_0.$$

Finally, let $\Psi^T=[\psi_1, \psi_2, \ldots, \psi_{NL}]=[0,0,1,0]$, we get

$$-0+0-\begin{bmatrix}K_{3,1} & K_{3,2} & K_{3,3} & K_{3,4}\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}+R_3=0\rightarrow\begin{bmatrix}K_{3,1} & K_{3,2} & K_{3,3}\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=R_3-K_{3,4}y_0.$$

Reason choosing $\Psi^T=[0,1,0,\ldots]$,

$$\begin{bmatrix}\psi_1 & \psi_2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}\psi_1 & \psi_2\end{bmatrix}\begin{bmatrix}p\\q\end{bmatrix}=\psi_1$$

$$\begin{matrix}\begin{bmatrix}\psi_1 & \psi_2\end{bmatrix}\rightarrow\begin{bmatrix}1 & 0\end{bmatrix}\\ \begin{bmatrix}\psi_1 & \psi_2\end{bmatrix}\rightarrow\begin{bmatrix}0 & 1\end{bmatrix}\end{matrix}\rightarrow\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}p\\q\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

This system equivalent to setting $\Psi^T=[1,0]$ & $\Psi^T=[0,1]$

Finite Element Method

Galerkin's approach for heat conduction

Finally, we get

$$\begin{bmatrix} (K_{11}+h) & K_{1,2} & K_{1,3} \\ K_{2,1} & K_{2,2} & K_{2,3} \\ K_{3,1} & K_{3,2} & K_{3,3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} R_1 + hy_\infty \\ R_2 \\ R_3 \end{bmatrix} - \begin{bmatrix} K_{1,4}y_0 \\ K_{2,4}y_0 \\ K_{3,4}y_0 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots \ (a)$$

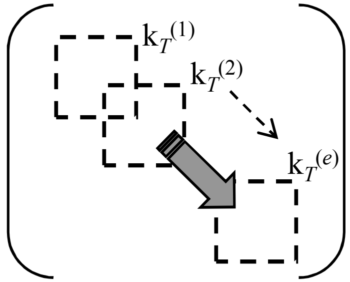
The element conductivity matrices are

$$\mathbf{k}_T^{(1)} = \frac{20}{0.3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{(1,\,2)}, \quad \mathbf{k}_T^{(2)} = \frac{30}{0.15}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{(2,\,3)}, \quad \mathbf{k}_T^{(3)} = \frac{50}{0.15}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{(3,\,4)}$$

The global $K_T=\Sigma\ k_T$ is obtained

$$\mathbf{K}_T = 66.7\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 8 & -5 \\ 0 & 0 & -5 & 5 \end{bmatrix}$$

$K_T =$



Since no heat generation Q occurs in this problem, we get $r_Q=[0\ 0]^T$, $R=[0\ 0\ 0]^T$.

Given $y_0=20$ºC, $y_\infty=800$ºC and $h=25$ W/m²·ºC, eq. (a) becomes

$$66.7\begin{bmatrix} 1.375 & -1 & 0 \\ -1 & 4 & -3 \\ 0 & -3 & 8 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0+25(800) \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -5(66.7)(20) \end{bmatrix} = \begin{bmatrix} 20{,}000 \\ 0 \\ 6670 \end{bmatrix}$$

This linear system can be solved using Thomas algorithm and we get $[y_1, y_2, y_3]=[304.6, 119.0, 57.1]$ ºC

$$\mathbf{Ax}=\mathbf{b},\ \ \mathbf{A}=\mathbf{LU} \rightarrow \begin{pmatrix} d_1 & e_1 & 0 & \cdots & 0 \\ c_2 & d_2 & e_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & c_{n-1} & d_{n-1} & e_{n-1} \\ 0 & \cdots & 0 & c_n & d_n \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ c_2 & \alpha_2 & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & c_{n-1} & \alpha_{n-1} & 0 \\ 0 & \cdots & 0 & c_n & \alpha_n \end{pmatrix} \cdot \begin{pmatrix} 1 & \beta_1 & 0 & \cdots & 0 \\ 0 & 1 & \beta_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & 1 & \beta_{n-1} \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

The whole Thomas algorithm can be summarized :

1. $\alpha_1=d_1$
2. $\alpha_i=d_i-c_i\beta_{i-1}$, $i=2,3,\ldots,n$
3. $\beta_i=e_i/\alpha_i$, $i=1,2,\ldots,n-1$.
4. $w_1=b_1/\alpha_1$
5. $w_i=(b_i-c_iw_{i-1})/\alpha_i$, $i=2,3,\ldots,n$.
6. $x_n=w_n$
7. $x_i=w_i-\beta_ix_{i+1}$, $i=n-1, n-2,\ldots,1$.

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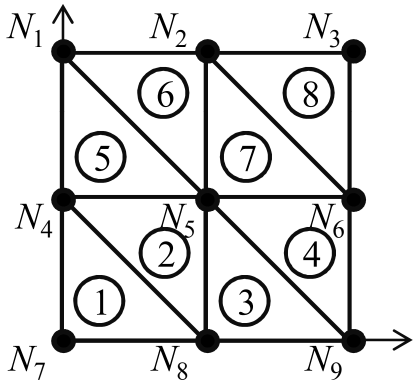
Preprocessing

Preprocessing of the problem includes one or more of the following tasks:

- Read geometry and material data (E), and boundary and initial conditions of the problem.
- Mesh generation.
- Generation of node numbers.
- Generation of coordinates and connectivity.



element	1	2	3	← local
1	7	8	4	
2	8	5	4	
3	8	9	5	↑
4	9	6	5	Global
5	4	5	1	
6	5	2	1	↓
7	5	6	2	
8	6	3	2	

Linear triangular element

Processing of FEM

Processing of the FEM includes one or more of the following tasks:

- Calculate element matrices.
- Assemble element equations.
- Solve the system of equations.

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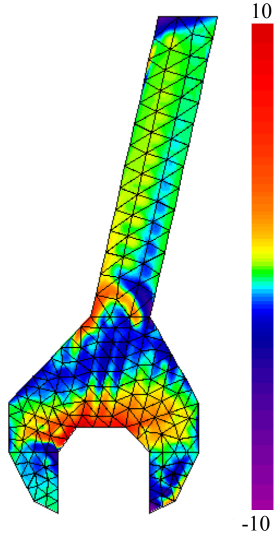
Postprocessing

Postprocessing of the FEM includes one or more of the following tasks:

- Computation of the primary and secondary variables at points of interest; primary variables are known at nodal points.
- Interpretation of the results to check whether the solution makes sense (based on physical Process and experience when other solutions are not available.
- Tabular and/or graphical presentation of the results. Contour plotting uses $\xi = \frac{2}{x_2 - x_1}(x - x_1) - 1$



Contour plot for stress

Interpolation of temperature within each element is given

$$y(\xi)=N_1 y_1+N_2 y_2 = \mathbf{N}\mathbf{y}^e$$

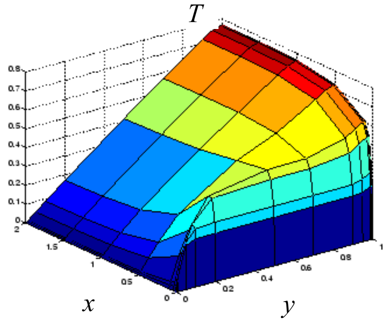
where $N_1=(1-\xi)/2$, $N_2=(1+\xi)/2$, ξ varies from -1 to +1, $\mathbf{N}=[N_1, N_2]$, $\mathbf{y}^e=[y_1, y_2]^T$.

The derivative of the solution is obtained by differentiation

Use chain rule, $$\frac{dy}{dx} = \frac{dy}{d\xi}\cdot\frac{d\xi}{dx} = \frac{2}{x_2 - x_1}\frac{d\mathbf{N}}{d\xi}\cdot\mathbf{y}^e = \frac{1}{l_e}[-1,1]\mathbf{y}^e = \mathbf{B}_T\mathbf{y}^e.$$

For element 1, we get $$\frac{dy^{e=1}}{dx} = \mathbf{B}_T\mathbf{y}^{e=1} = \frac{1}{l_e}[-1,1]\mathbf{y}^{e=1} = \frac{1}{0.3}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}304.6\\119.0\end{bmatrix} = -618.67$$

For element 2, we get $$\frac{dy^{e=2}}{dx} = \mathbf{B}_T\mathbf{y}^{e=2} = \frac{1}{l_e}[-1,1]\mathbf{y}^{e=2} = \frac{1}{0.15}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}119.0\\57.1\end{bmatrix} = -412.67$$



Contour plot for $T(x,y)$

Note that the derivative above is discontinuous, for any order element, at the nodes connecting the different elements because the continuity of the derivative of FE solution at the connecting nodes is not imposed.

References

- Brian Bradie, A Friendly Introduction to Numerical Analysis, Prentice Hall, New Jersey, 2006.
- Fausett L.V, Numerical Methods; algorithm and applications, Prentice Hall, New Jersey,2003
- Rao S.S, Applied Numerical Methods for Engineers and Scientist, Prentice Hall, New Jersey,2002
- Faires J.D. Burden R, Numerical Methods, 2nd Edition, Thomson Brooks/Cole, Australia,1998
- Burden R.L, Faires J.D & Reynolds A.C, Numerical Analysis, 5th edition, PWS-KENT Pub, Boston, 1993
- Introduction to Finite Elements in Engineering , T. R. Chandrupatla, A. D. Belegundu, 3rd edition, Prentice Hall, N.J. 2002