Numerical Methods II

SSCM 3423

Chapter 4

This chapter solves boundary value problems (BVP) using shooting method

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Shooting method – linear problem

BVP - Dirichlet Boundary condition

$$y'' = p(x)y' + q(x)y + r(x),$$
$$y(a) = \alpha, \quad y(b) = \beta.$$

Let IVP1 (initial value problem)

$$\left.\begin{aligned} &y'' = p(x)y' + q(x)y + r(x), \\ &y(a) = \alpha, \quad y'(a) = 0. \end{aligned}\right\} \quad \text{Solution: } y_1(x)$$

Let IVP2 (initial value problem) (homogeneous)

$$\left.\begin{aligned} &y'' = p(x)y' + q(x)y, \\ &y(a) = 0, \quad y'(a) = 1. \end{aligned}\right\} \quad \text{Solution: } y_2(x)$$

Linearity of De: $y = y_1(x) + cy_2(x)$ is the solution for $y'' = p(x)y' + q(x)y + r(x)$.

⇒ $y(a) = y_1(a) + cy_2(a) = \alpha + c \cdot 0 = \alpha$ ⇒ $y(b) = y_1(b) + cy_2(b) = \beta$

⇒ $$c = \frac{\beta - y_1(b)}{y_2(b)}$$

Shooting method – linear problem

BVP - Dirichlet Boundary condition

Optimal RK2 method, $y'=f(x,y)$

$$y_{i+1} \approx y_i + \frac{1}{4}(K_1 + 3K_2)$$
$$K_1 = hf(t_i, y_i)$$
$$K_2 = hf\left(t_i + \frac{2}{3}h, y_i + \frac{2}{3}K_1\right)$$

$$-u''+\pi^2 u = 2\pi^2 \sin(\pi x), \quad u(0)=\alpha=0,\ u(1)=\beta=0.$$ BVP

General solution: $ae^{-\pi x} + be^{\pi x} + \sin(\pi x)$

$$v''=\pi^2 v - 2\pi^2 \sin(\pi x), \quad v(0)=\alpha=0,\ v'(0)=0.$$ IVP1, solution: v

$$w''=\pi^2 w, \quad w(0)=0,\ w'(0)=1.$$ IVP2, (homogeneous) solution: w

To system

$$\frac{d}{dx}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ \pi^2 v_1 - 2\pi^2 \sin(\pi x) \end{bmatrix}$$

To system

$$\frac{d}{dx}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_2 \\ \pi^2 w_1 \end{bmatrix}$$

Solve using RK4

$y(x+h) \approx y(x) + 1/6[\, k_1+2k_2+2k_3+k_4]$

$k_1=hf(x,y)$, $k_2=hf(x+ ½h, y+ ½\, k_1)$,

$k_3=hf(x+ ½h, y+ ½k_2)$, $k_4=hf(x+h, y+k_3)$

x_i	$v_i(x_i)$	$w_i(x_i)$
0.00	0.00000	0.000000
0.25	-0.157372	0.275702
0.50	-1.290357	0.730213
0.75	-4.490694	1.657343
1.00	-11.466375	3.656793

Not satisfy right BC!

Shooting method – linear problem

BVP - Dirichlet Boundary condition

$$y'' = p(x)y' + q(x)y + r(x),$$
$$y(a) = \alpha, \quad y(b) = \beta.$$

$$\left.\begin{aligned} -u'' + \pi^2 u = 2\pi^2 \sin(\pi x), \\ u(0) = u(1) = 0. \end{aligned}\right\} \text{BVP}$$

$y = y_1(x) + c y_2(x)$ is the solution for $y'' = p(x)y' + q(x)y + r(x)$.

$$c = \frac{\beta - y_1(b)}{y_2(b)}$$

$$c = \frac{\beta - v(b)}{w(b)} = \frac{0 - (-11.466375)}{3.656793} = 3.135637$$

$y_i = y_1(x_i) + c y_2(x_i) = v(x_i) + 3.135637 w(x_i)$

Solve using RK4

x_i	$v_i(x_i)$	$w_i(x_i)$	y_i	***Exact, sin(πx)***
0.00	0.00000	0.000000	0.00000	0.00000
0.25	-0.157372	0.275702	0.707129	0.707107
0.50	-1.290357	0.730213	0.999327	1.00000
0.75	-4.490694	1.657343	0.706132	0.707107
1.00	-11.466375	3.656793	0.000000	0.000000

Optimal RK2 method, $y' = f(x,y)$

$$y_{i+1} \approx y_i + \frac{1}{4}(K_1 + 3K_2)$$
$$K_1 = hf(t_i, y_i)$$
$$K_2 = hf\left(t_i + \frac{2}{3}h, y_i + \frac{2}{3}K_1\right)$$

Shooting method – linear problem

BVP – Left: Dirichlet BC, right: Robin BC

$$y'' = p(x)y' + q(x)y + r(x),$$
$$y(a) = \alpha,\ \ \beta_1 y(b) + \beta_2 y'(b) = \beta_3.$$

Let IVP1 (initial value problem)

$$\left.\begin{aligned} &y'' = p(x)y' + q(x)y + r(x),\\ &y(a) = \alpha,\ \ y'(a) = 0. \end{aligned}\right\}\quad \text{Solution: } y_1(x)$$

Let IVP2 (initial value problem) (homogeneous)

$$\left.\begin{aligned} &y'' = p(x)y' + q(x)y,\\ &y(a) = 0,\ \ y'(a) = 1. \end{aligned}\right\}\quad \text{Solution: } y_2(x)$$

Linearity of DE: $y = y_1(x) + cy_2(x)$ is the solution for $y''=p(x)y'+q(x)y+r(x)$.

⇒ $y(a) = y_1(a) + cy_2(a) = \alpha + c\cdot 0 = \alpha$ ⇒ $\beta_1[y_1(b)+cy_2(b)]+\beta_2[y'_1(b)+cy'_2(b)] = \beta_3$

⇒ $$c = \frac{\beta_3 - \beta_1 y_1(b) - \beta_2 y_1'(b)}{\beta_1 y_2(b) + \beta_2 y_2'(b)}$$

Shooting method – linear problem

BVP – Left: Dirichlet BC, right: Neumann BC

$$y'' = p(x)y' + q(x)y + r(x),$$
$$y(a) = \alpha, \quad y'(b) = \beta \, .$$

Let IVP1 (initial value problem)

$$y'' = p(x)y' + q(x)y + r(x),$$
$$y(a) = \alpha, \quad y'(a) = 0.$$

Solution: $y_1(x)$

Let IVP2 (initial value problem) (homogeneous)

$$y'' = p(x)y' + q(x)y,$$
$$y(a) = 0, \quad y'(a) = 1.$$

Solution: $y_2(x)$

Linearity of DE: $y= y_1(x)+cy_2(x)$ is the solution for $y''=p(x)y'+q(x)y+r(x)$.

⇒ $y(a)= y_1(a)+cy_2(a)=\alpha+c\cdot 0= \alpha$ ⇒ $[y'_1(b)+cy'_2(b)]= \beta$

⇒ $$c = \frac{\beta - y_1'(b)}{y_2'(b)}$$

Shooting method – linear problem

BVP – Left: Neumann BC, right: Dirichlet BC (or others)

$$y'' = p(x)y' + q(x)y + r(x),$$
$$y'(a) = \alpha, \quad y(b) = \beta \ .$$

Let IVP1 (initial value problem)

$$\left.\begin{aligned} & y'' = p(x)y' + q(x)y + r(x), \\ & y(a) = 0, \quad y'(a) = \alpha. \end{aligned}\right\} \quad \text{Solution: } y_1(x)$$

Let IVP2 (initial value problem) (homogeneous)

$$\left.\begin{aligned} & y'' = p(x)y' + q(x)y, \\ & y(a) = 1, \quad y'(a) = 0. \end{aligned}\right\} \quad \text{Solution: } y_2(x)$$

Linearity of DE: $y = y_1(x) + cy_2(x)$ is the solution for $y'' = p(x)y' + q(x)y + r(x)$.

⇒ $y'(a) = y'_1(a) + cy'_2(a) = \alpha + c \cdot 0 = \alpha$ ⇒ $y(b) = y_1(b) + cy_2(b) = \beta$

⇒ $$c = \frac{\beta - y_1(b)}{y_2(b)}$$

Shooting method – linear problem

BVP – Left: Neumann BC, right: Robin BC

$$u'' + u = \sin(3x),$$
$$u'(0) = \alpha = 1,\ u(\pi/2) + u'(\pi/2) = \beta = -1.$$

Exact solution: $u = a\sin(x) + b\cos(x) - 1/8\sin(3x)$

Let IVP1 (initial value problem)

$$v'' + v = \sin(3x),$$
$$v(0) = 0,\ v'(0) = \alpha = 1.$$

Solution: $v(x)$

$$\frac{d}{dx}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ \sin(3x) - v_1 \end{bmatrix}$$

Let IVP2 (initial value problem) (homogeneous)

$$w'' + w = 0,$$
$$w(0) = 1,\ w'(0) = 0.$$

Solution: $w(x)$

$$\frac{d}{dx}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_2 \\ -w_1 \end{bmatrix}$$

Linearity of DE: $u = v(x) + cw(x)$ is the solution for $u'' + u = sin(3x)$.

$$u'(0) = v'(0) + cw'(0) = \alpha + c\cdot 0 = \alpha = 1$$

$$u(\pi/2) + u'(\pi/2) = v(\pi/2) + v'(\pi/2) + cw(\pi/2) + cw'(\pi/2) = \beta = -1$$

$$c = \frac{\beta - v(\pi/2) - v'(\pi/2)}{w(\pi/2) + w'(\pi/2)}$$

Optimal RK2 method, $y' = f(x,y)$

$$y_{i+1} \approx y_i + \frac{1}{4}(K_1 + 3K_2)$$
$$K_1 = hf(t_i, y_i)$$
$$K_2 = hf\left(t_i + \frac{2}{3}h, y_i + \frac{2}{3}K_1\right)$$

Shooting method – linear problem

BVP – Left: Neumann BC, right: Robin BC

$$u'' + u = \sin(3x),$$
$$u'(0) = \alpha = 1,\ u\left(\pi/2\right) + u'\left(\pi/2\right) = \beta = -1.$$

general solution: $u = a\sin(x) + b\cos(x) - 1/8\,\sin(3x)$

$$c = \frac{\beta - v\left(\pi/2\right) - v'\left(\pi/2\right)}{w\left(\pi/2\right) + w'\left(\pi/2\right)} = \frac{-1 - 1.499586 - 0.000195}{0.000294 - 0.999900} = 2.500766$$

$u_i = v(x_i) + cw(x_i) = v(x_i) + 2.500766\,w(x_i)$

Solve using RK4

Exact solution: $u = (11/8)\sin(x) + (5/2)\cos(x) - 1/8\,\sin(3x)$

x_i	$v(x_i)$	$v'(x_i)$	$w(x_i)$	$w'(x_i)$	*Approx, $u(x_i)$*	*Exact, $u(x_i)$*
0.00	0.00000	1.000000	1.00000	0.00000	2.500766	2.5
π/8	0.411165	1.126997	0.923885	-0.382606	2.721585	2.720404
π/4	0.884311	1.237806	0.707176	-0.706967	2.652793	2.65165
3π/8	1.318095	0.873002	0.382859	-0.923726	2.275536	2.274878
π/2	1.499586	0.000195	0.000294	-0.999900	1.500321	1.5

Optimal RK2 method, $y' = f(x,y)$

$$y_{i+1} \approx y_i + \frac{1}{4}\left(K_1 + 3K_2\right)$$
$$K_1 = hf\left(t_i, y_i\right)$$
$$K_2 = hf\left(t_i + \frac{2}{3}h, y_i + \frac{2}{3}K_1\right)$$

Shooting method – linear problem

BVP – Left: Robin BC, right: Dirichlet BC

$$y'' = p(x)y' + q(x)y + r(x),$$
$$\alpha_1 y(a) + \alpha_2 y'(a) = \alpha_3, y(b) = \beta.$$

Let IVP1 (initial value problem)

$$y'' = p(x)y' + q(x)y + r(x),$$
$$y(a) = 0, \quad y'(a) = 0.$$

Solution: $y_1(x)$

Let IVP2 (initial value problem) (homogeneous)

$$y'' = p(x)y' + q(x)y,$$
$$y(a) = 1, \quad y'(a) = 0.$$

Solution: $y_2(x)$

Let IVP3 (initial value problem)

$$y'' = p(x)y' + q(x)y,$$
$$y(a) = 0, \quad y'(a) = 1.$$

Solution: $y_3(x)$

Linearity of DE: $y = y_1(x) + c_1 y_2(x) + c_2 y_3(x)$ is the solution for $y'' = p(x)y' + q(x)y + r(x)$.

$\alpha_1 y(a) + \alpha_2 y'(a) = \alpha_3$ ⇒ $[\alpha_1 y_2(a) + \alpha_2 y'_2(a)]c_1 + [\alpha_1 y_3(a) + \alpha_2 y'_3(a)]c_2 = \alpha_3 - \alpha_1 y_1(a) - \alpha_2 y'_1(a)$

⇒ $\alpha_1 c_1 + \alpha_2 c_2 = \alpha_3$

$y(b) = \beta$ ⇒ $y_1(b) + c_1 y_2(b) + c_2 y_3(b) = \beta$ ⇒ System of 2 equations!

Shooting method – nonlinear problem

BVP - Dirichlet Boundary condition

$$yy'' + (y')^2 + 1 = 0,$$
$$y(1) = 1,\ \ y(2) = 2.$$

$$yy'' + (y')^2 + 1 = 0,$$
$$y(1) = 1.$$

General solution

$$y = \sqrt{2a - 2xa + 2x - x^2}$$

Let IVP (initial value problem)

$$yy'' + (y')^2 + 1 = 0,$$
$$y(1) = 1,\ \ y'(1) = p = p_i.$$

General solution

$$y = \sqrt{-2p + 2xp + 2x - x^2}$$

Solution: $y(x;p)$

$$\frac{d}{dx}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ -1-(y_2)^2 / y_1 \end{bmatrix}$$

Get p^*, $\rightarrow$ $y(2;p^*)=2$.

p_0 & p_1 $\rightarrow$ arbitrary value

Rootfinding, objective function, $\rightarrow F(p)=2-y(2;p)$.

$$F(p_n) \approx 0$$

Avoid using Newton's method $\rightarrow$ $F' = \frac{d}{dp} y(2;p)$

Use Secant method $\rightarrow$ $p_n = p_{n-1} - F(p_{n-1})\dfrac{p_{n-1} - p_{n-2}}{F(p_{n-1}) - F(p_{n-2})}$

Start with p_0 & p_1 $\rightarrow$ get p_2 …

$$p_2 = p_1 - F(p_1)\frac{p_1 - p_0}{F(p_1) - F(p_0)}$$
$$= 1 - 0.585803\frac{1-0}{0.585803 - 1.895899} = 1.447145$$

Use RK4 $\rightarrow p_0=0=y'(1)$

i	p_i	$y(2;p_i)$	$F(p_i)$ (error)
0	0.000000	0.104101	1.895899
1	1.0	1.414197	0.585803
2	1.447145	1.701210	0.298790
3	1.912638	1.955692	0.044308

Shooting method – nonlinear problem

BVP - Dirichlet Boundary condition

$$yy'' + (y')^2 + 1 = 0,$$
$$y(1) = 1,\ \ y(2) = 2.$$

IVP (initial value problem)

$$yy'' + (y')^2 + 1 = 0,$$
$$y(1) = 1,\ \ y'(1) = p = p_i.$$

Solution: $y(x;p)$

General solution

$$y = \sqrt{-2p + 2xp + 2x - x^2}$$

$$yy'' + (y')^2 + 1 = 0,$$
$$y(1) = 1.$$

General solution

$$y = \sqrt{2a - 2xa + 2x - x^2}$$

Rootfinding, objective function, $\rightarrow F(p)=2-y(2;p)$.

$F(p_n) \approx 0$ Use Secant method

$$p_n = p_{n-1} - F(p_{n-1})\frac{p_{n-1} - p_{n-2}}{F(p_{n-1}) - F(p_{n-2})}$$

Use RK4

$\rightarrow p_0=0=y'(1)$

i	p_i	$y(2;p_i)$	$F(p_i)$ (error)
3	1.912638	1.955692	0.044308
4	1.993685	1.996677	0.003322
5	2.000256	1.999963	3.698×10^{-5}
6	2.000330	1.999999969	3.088×10^{-8}

x_i	$y(x_i;p_6)$	$y(x_i)$ exact	Absolute error
1.00	1.000000	1.000000	
1.10	1.178956	1.178983	0.000027
1.20	1.326623	1.326650	0.000027
1.30	1.452560	1.452584	0.000024

Shooting method – nonlinear problem

BVP – Left Neumann, right Robin BC

$$2xy'' + (y')^2 - 4y = 4x,$$
$$y'(1) = 4,\ \ y(3) + 2y'(3) = 32.$$

Exact solution
$y(x)=(x+1)^2$

IVP (initial value problem)

$$2xy'' + (y')^2 - 4y = 4x,$$
$$y'(1) = 4,\ \ y(1) = p = p_i$$

Solution: $y(x;p)$

$$y_1 = y(x),\quad y_2 = y'(x).$$

$$\frac{d}{dx}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ \left(4x + 4y_1 - (y_2)^2\right)/2x \end{bmatrix}$$

Rootfinding, objective function, $\rightarrow F(p)=32 - y(3;p) - 2y'(3;p)$.

Use RK4
$\rightarrow p_0=0=y'(1)$

$F(p_n) \approx 0$ Use Secant method

$$p_n = p_{n-1} - F(p_{n-1})\frac{p_{n-1} - p_{n-2}}{F(p_{n-1}) - F(p_{n-2})}$$

i	p_i	$y(3;p_i)$	$y'(3;p_i)$	$F(p_i)$ (error)
1	0	8.303482	5.781270	12.133978
2	1	10.361601	6.445059	8.748282
3	3.583894	15.254059	7.811455	1.123031
4	3.964445	15.936660	7.984070	0.0952008
5	3.999693	15.999486	7.999773	9.677×10^{-4}

$\Rightarrow$ $p_6=4.000055$

x_i	$y(x_i;p_6)$	$y(x_i)$ exact	Absolute error
1.00	4.000055	4.00	0.000055
1.20	4.840102	4.84	0.000102
1.40	5.760118	5.76	0.000118
…	…	…	…
3.00	16.000132	16.00	0.000132

Try this!

$$2xy'' + (y')^2 - 4y = 4x,$$
$$y'(2) = 6,\ \ y'(2.1) = 6.2$$

$$y'' = -2yy',\ 0 \le x \le 1,$$
$$y(0) = 1,\ \ y(0.2) = 0.8333$$

Exact solution: $y=1/(x+1)$.

Optimal RK2 method, $y'=f(x,y)$

$$y_{i+1} \approx y_i + \frac{1}{4}(K_1 + 3K_2)$$
$$K_1 = hf(t_i, y_i)$$
$$K_2 = hf\left(t_i + \frac{2}{3}h,\ y_i + \frac{2}{3}K_1\right)$$

Shooting method – nonlinear problem

BVP – Left Robin, right Dirichlet

$$y'' = 3y^3,$$

$$3y(0) - 9y'(0) = 2, \quad y(1) = \frac{1}{4}$$

IVP (initial value problem)

$$y'' = 3y^3,$$

$$3y(0) - 9p_i = 2, \quad y'(0) = p_i$$

IVP (initial value problem)

$$y'' = 3y^3,$$

$$3p_i - 9y'(0) = 2, \quad y(0) = p_i$$

Try this!!

$$y'' = -2yy', \; 0 \le x \le 1,$$

$$y(0) = 1, \quad y(0.2) = 0.8333$$

Exact solution: $y=1/(x+1)$.

$$\mathbf{y} = \begin{bmatrix} y \\ y' \end{bmatrix} \rightarrow \frac{d}{dx}\mathbf{y} = \frac{d}{dx}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ -2y_1y_2 \end{bmatrix}$$

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