SSCM 1023 MATHEMATICAL METHODS I

TOPIC: INTEGRATION

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INTEGRATION

Recall: Methods involved:

- Substitution of *u*
- By parts
- Tabular method
- Partial fractions
- Trigonometric substitutions

4.1 Integrals of Hyperbolic Functions

Table of Integration for Hyperbolic Functions

1. $\int \sinh x dx = \cosh x + C$
2. $\int \cosh x dx = \sinh x + C$
3. $\int \sec h^2 x dx = \tanh x + C$
4. $\int \cos \mathrm{ech}^2 \mathrm{xdx} = -\coth \mathrm{x} + \mathrm{C}$
5. $\int \sec hx \tanh x dx = -\sec hx + C$
6. $\int \cos echx \coth x dx = -\cos echx + C$

Example 1:

Integrate the following hyperbolic functions using appropriate technique (definition, identities, etc) and method (substitution, by parts, tabular, etc).

a) $\int \sinh 2x \cosh 3x \, dx$

b) $\int \frac{\cosh x}{2 + 3\sinh x} dx$

c) $\int \sinh^3 x \, dx$

d) $\int x \cosh 2x \, dx$

e) $\int \sinh\left(\frac{x}{2}\right) \cosh\left(\frac{x}{2}\right) dx$

f) $\int \sqrt{\tanh x} \, \mathrm{sec}\, h^2 x \, dx$

4.2 Integration of Inverse Trigonometric Functions

Integration formulae of the Inverse Trigonometric Functions

Example 2 :

1. Evaluate the following integrals

a) $\int_0^1 \tan^{-1} x \, dx$

b) $\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}} \, dx$

c) $\int \frac{\sqrt{\tan^{-1} x}}{1+x^2} \, dx$

2. Use partial fraction decomposition to solve

$$\int_0^1 \frac{x^2-2x}{(2x+1)(x^2+1)} \, dx \, .$$

Differentiation	Integration
$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$	$\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x+C$
$\frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}}$	$\int\frac{-dx}{\sqrt{1-x^2}}=\cos^{-1}x+C$
$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$	$\int\frac{dx}{1+x^2}=\tan^{-1}x+C$
$\frac{d}{dx}(\cot^{-1}x)=\frac{-1}{1+x^2}$	$\int\frac{-dx}{1+x^2}=\cot^{-1}x+C$
$\frac{d}{dx}(\sec^{-1}x)=\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\int\frac{dx}{\lvert x\rvert\sqrt{x^2-1}}=\sec^{-1}x+C$
$\frac{d}{dx}(\csc^{-1}x)=\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	$\int\frac{-dx}{\lvert x\rvert\sqrt{x^2-1}}=\csc^{-1}x+C$

Example 3 : Evaluate the following integrals

1. a) $\int \frac{dx}{\sqrt{16-x^2}}$

   b) $\int \frac{2\,dx}{3+x^2}$

2. a) $\int \frac{dx}{\sqrt{1-4x^2}}$

   b) $\int \frac{dx}{4+3x^2}$

3. Use completing the square technique to solve:

a) $\int \frac{dx}{\sqrt{-x^2+2x+3}}$

b) $\int \frac{dx}{x^2-2x+2}$

4. By using substitution $t=\tan\left(\frac{x}{2}\right)$, show that

$$\int \frac{dx}{5+4\cos x} = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\left(\frac{x}{2}\right)\right)+C$$

4.3 Integration involving Inverse Hyperbolic Functions

Integration formulae of the Inverse Hyperbolic Functions:

Differentiation	**Integration**
$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$	$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C$
$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$	$\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + C$
$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$	$\int \frac{dx}{1-x^2} = \tanh^{-1} x + C$

Example 4: Solve the following:

a) $\int \frac{dx}{\sqrt{3x^2+2}}$

b) $\int \frac{dx}{\sqrt{2(x-3)^2+1}}$

c) $\int \frac{dx}{\sqrt{x^2+4x+3}}$

2. Show that $\int \frac{x+1}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1} + \sinh^{-1} x + C$.

4.4 Further Applications of Integrations

4.4.1 a) Arc Length in Parametric Form

The length of the parametric curve $(x(t), y(t))$ as t varies from t_0 to t_1 is given by

$$\mathcal{L} = \int_{t=t_0}^{t=t_1} \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

Example 5:

Consider the curve given by $x(t) = \cos t$, $y(t) = \sin t$, $0 \le t \le \pi$. Find the length of the curve. (ans: pi)

Its length is:

4.4.1 b) Arc Length in Cartesian Form

If we wish to find the length of a Cartesian curve which is the graph of a function

$$y = f(x),\ a \le x \le b,$$

we let

$$x(t) = t, \quad y(t) = f(x(t)) = f(x)$$

and we get

$$x'(t) = 1 \text{ and } y'(t) = f'(x(t))x'(t) = f'(x),$$

therefore we have a simple formula for the length:

$$\mathcal{L} = \int_{x=a}^{x=b} \sqrt{1 + (f'(x))^2}\,dx = \int_a^b \sqrt{1 + (f'(x))^2}\,dx = \int_a^b \sqrt{1 + (y')^2}\,dx$$

Similarly, if we have a curve $x = g(y),\ c \le y \le d$, we get

$$\mathcal{L} = \int_{y=c}^{y=d} \sqrt{1 + (g'(y))^2}\,dy = \int_c^d \sqrt{1 + (g'(y))^2}\,dy = \int_c^d \sqrt{1 + (x')^2}\,dy$$

Example 6:

Find the length of the curve

a) $y = \frac{1}{3}(x^2 + 2)^{\frac{3}{2}}, 0 \le x \le 3.$ (ans:12)

b) $x = \frac{2}{3}(y - 1)^{\frac{3}{2}}, 1 \le y \le 4.$ (ans:14/3)

Example 7:

Find the length of the arc of the parabola $y^2 = x$ from (0, 0) to (1, 1).

Ans: $L = \frac{\sqrt{5}}{2} + \frac{\ln(\sqrt{5} + 2)}{4}$

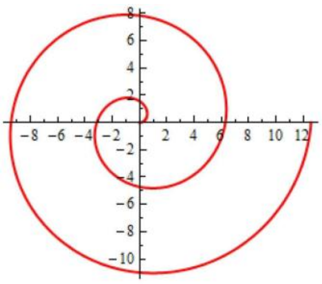
4.4.2 Arc Length in Polar Coordinates

The length of a curve with polar equation $r = f(\theta)$, $a \leq \theta \leq b$, is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example 8:

a) Find the length of the curve $r = \theta$, $0 \leq \theta \leq 1$.



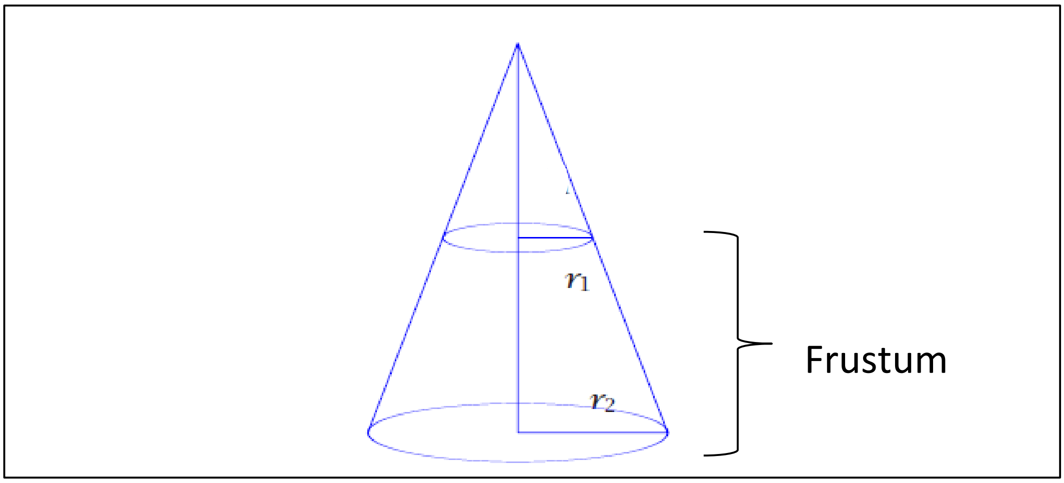
$\frac{1}{2}(\sqrt{2} + \ln(1 + \sqrt{2}))$

b) Find the length of the cardioid $r = 1 - \cos\theta$, $0 \leq \theta \leq 2\pi$.

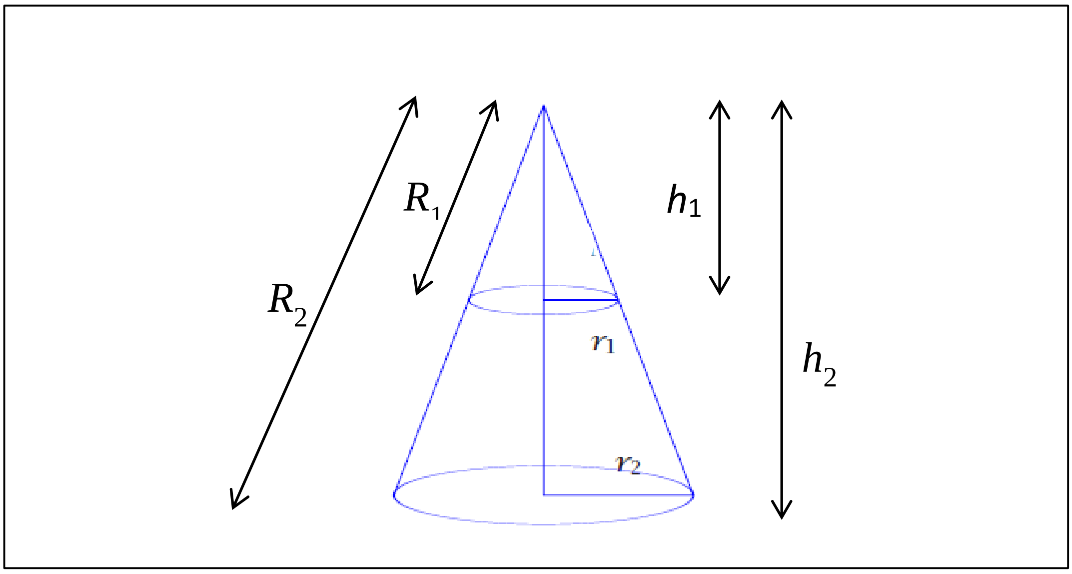
Ans: 8

4.4.3 Area of Surface of Revolution in Cartesian Form

Consider two cones, with one being a subset of the other; we can calculate the area of the region between the bases of the two cones. This region is called a **frustum**.



Let the larger and smaller cones have heights and radii h_2 and r_2 and h_1 and r_1.



It is clear that $R_1 = \sqrt{r_1^2 + h_1^2}$ and $R_2 = \sqrt{r_2^2 + h_2^2}$. Therefore,

area of larger cone, $A_2 = \pi r_2 R_2 = \pi r_2 \sqrt{r_2^2 + h_2^2}$,

area of smaller cone, $A_1 = \pi r_1 R_1 = \pi r_1 \sqrt{r_1^2 + h_1^2}$.

The area of the frustum, thus,

$$\begin{aligned} A &= A_2 - A_1 \\ &= \pi r_2 \sqrt{r_2^2 + h_2^2} - \pi r_1 \sqrt{r_1^2 + h_1^2} \\ &= \pi \left[r_2 \sqrt{r_2^2 + h_2^2} - r_1 \sqrt{r_1^2 + h_1^2} \right] \\ &= \pi (r_2 R_2 - r_1 R_1) \\ &= 2\pi r R \quad \text{where } R = R_2 - R_1 \text{ and } r = \frac{r_1 + r_2}{2} \end{aligned}$$

We can then use this formula to derive a formula for the area of the surface obtained by rotating the curve $(x(t), y(t))$, $t_1 \le t \le t_2$ about the x- and $y-$axes respectively:

$$S_x = \int_{t_1}^{t_2} 2\pi y(t) \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

$$S_y = \int_{t_1}^{t_2} 2\pi x(t) \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

If the curve is the graph of a function $y = f(x)$, $a \le x \le b$, then the area of the surface obtained by revolving the curve about the x-axis is

$$S_x = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx$$

and the area of the surface obtained by revolving the curve about the y-axis is

$$S_y = \int_a^b 2\pi x\sqrt{1 + (f'(x))^2}\,dx$$

If the curve is the graph of a function $x = g(y)$, $c \le x \le d$, then the area of the surface obtained by revolving the curve about the x-axis is

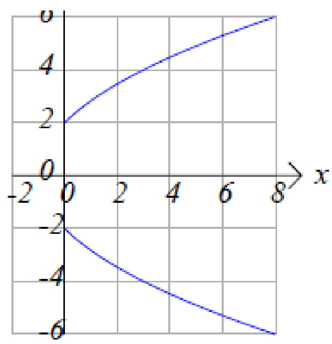
$$S_x = \int_c^d 2\pi y\sqrt{1 + (g'(y))^2}\,dy$$

and the area of the surface obtained by revolving the curve about the y-axis is

$$S_y = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\,dy$$

Example 9:

a) Find the area of the surface obtained by rotating the curve $y^2 = 4x + 4$, $0 \le x \le 8$, about the x-axis.



Ans: $\frac{16\pi}{3}\left(5\sqrt{5} - 1\right)\sqrt{2}$

b) Find the area of the surface obtained by rotating the curve $x = 1 + 2y^2$, $1 \le y \le 2$, about the x-axis.

Ans: $\frac{\pi}{24}\left(65\sqrt{65} - 17\sqrt{17}\right)$

4.4.4 Area of a Surface of Revolution in Polar Form

The areas of the surfaces generated by revolving the curve $r = f(\theta),\ a \le \theta \le b$ about the *x*- and *y*-axis are given by the following formulas:

- Revolution about *x*-axis, $(y \ge 0)$:

$$S_x = \int_a^b 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

- Revolution about *y*-axis, $x \ge 0$:

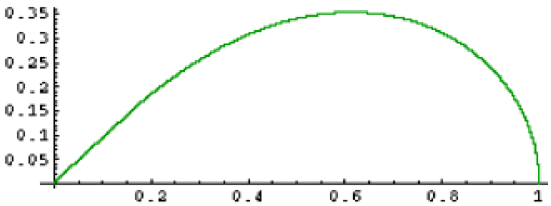
$$S_y = \int_a^b 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

Example 10:

Find the area of the surface generated by revolving

$$r = \sqrt{\cos 2\theta},\ 0 \le \theta \le \frac{\pi}{4}$$

about the *x*-axis.



Ans: $2\pi - \frac{2\pi}{\sqrt{2}}$

Summary Formula for Area of Revolution:

Type of Equation	Revolve about x-axis	Revolve about y-axis
Parametric $x = f(t)$, $y = g(t)$	$S_x = \int_{t_1}^{t_2} 2\pi y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$	$S_y = \int_{t_1}^{t_2} 2\pi x(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$
$y = f(x)$	$S_x = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx$	$S_y = \int_a^b 2\pi x\sqrt{1 + (f'(x))^2}\,dx$
$x = g(y)$	$S_x = \int_c^d 2\pi y\sqrt{1 + (g'(y))^2}\,dy$	$S_y = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\,dy$
Polar form $r = f(\theta)$	$S_x = \int_a^b 2\pi r\sin\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$	$S_y = \int_a^b 2\pi r\cos\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$

4.5 Appendix

1. Partial fraction decomposition.

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$,
	where x^2+bx+c cannot be factorised further	

2. Integrations involving $\sqrt{Ax^2+Bx+C}$

Expression	Substitution
$\sqrt{x^2+k^2}$	$x=k\tan\theta$ or $x=k\sinh\theta$
$\sqrt{x^2-k^2}$	$x=k\sec\theta$ or $x=k\cosh\theta$
$\sqrt{k^2-x^2}$	$x=k\sin\theta$ or $x=k\tanh\theta$

4.6 References

1. George B. Thomas, Maurice D. Weir, Joel R. Hass, and Frank R. Giordano. 2005. *Thomas' Calculus Early Transcendental (11th Edition) (Thomas Series)*. Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA.

2. Abdul Wahid Md Raji, Hamisan Rahmat, Ismail Kamis, Mohd Nor Mohamad, Ong Chee Tiong. ***Engineering mathematics I***, Penerbit UTM, 2012.