SSCM 1023 MATHEMATICAL METHODS I

TOPIC: DIFFERENTIATION

SHAZIRAWATI MOHD PUZI

&

NORZIEHA MUSTAPHA

DEPARTMENT OF MATHEMATICAL SCIENCES, UTM JB

DIFFERENTIATION

3.1 Differentiation of hyperbolic functions

3.2 Differentiation of inverse trigonometric functions

3.3 Differentiation of inverse hyperbolic functions

3.4 References

***Recall: Methods of differentiation**

-Chain rule
-Product differentiation
-Quotient differentiation
-Implicit differentiation

3.1 Differentiation of Hyperbolic Functions

***Recall:* Definition:**

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}$$

$$\text{cosech}\, x = \frac{1}{\sinh x}$$

$$\text{sech}\, x = \frac{1}{\cosh x}$$

Derivatives of hyperbolic functions

Example 3.1: Find the derivatives of

(a) sinh x (b) cosh x (c) tanh x

Solution:

(a) $$\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{1}{2}(e^x + e^{-x}) = \cosh x$$

(b) $$\frac{d}{dx}\cosh x = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{1}{2}(e^x - e^{-x}) = \sinh x$$

(c) $\frac{d}{dx}\tanh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$

Using quotient diff:

$$= \frac{\left(e^x + e^{-x}\right)\left(e^x + e^{-x}\right) - \left(e^x - e^{-x}\right)\left(e^x - e^{-x}\right)}{\left(e^x + e^{-x}\right)^2}$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{\left(e^x + e^{-x}\right)^2}$$

$$= \frac{4}{\left(e^x + e^{-x}\right)^2} \quad = \left(\frac{2}{\left(e^x + e^{-x}\right)}\right)^2$$

$$= \left(\frac{1}{\cosh x}\right)^2 \quad = \text{sech}^2 x$$

Using the same methods, we can obtain the derivatives of the other hyperbolic functions and these gives us the standard derivatives as shown in the following table.

Standard Derivatives of Hyperbolic Functions

$y = f(x)$	$\dfrac{dy}{dx} = f'(x)$
$\cosh x$	$\sinh x$
$\sinh x$	$\cosh x$
$\tanh x$	$\text{sech}^2 x$
$\text{sech}\, x$	$-\text{sech}\, x \tanh x$
$\text{cosech}\, x$	$-\text{cosech}\, x \coth x$
$\coth x$	$-\text{cosech}^2 x$

Example 3.2:

1.Find the derivatives of the following functions:

a) $y = \frac{x^3}{\sinh x}$

b) $y = \tanh^3 2x$

c) $y = e^{\cosh 4x^2}$

2. Find the derivatives of the following functions:

(a) $y = \cosh(3x)$

(b) $r = \sinh(2t^2 - 1)$

(c) $g(x) = (x-1)^3 \operatorname{sech}^2 x$

(d) $y = \tanh(\ln x)$

3.(Implicit differentiation)

Find $\frac{dy}{dx}$ from the following expressions:

(a) $x = y^2 \sinh 4x + \cosh y$

(b) $y = \tanh(x + y)$

3.2 Differentiation Involving Inverse Trigonometric Functions

Recall: Definition of inverse trigonometric functions

Function	Domain	Range
$\sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$\cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$\tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$\sec^{-1} x$	$\lvert x \rvert \ge 1$	$0 \le y < \frac{\pi}{2} \cup \frac{\pi}{2} < y < \pi$
$\cot^{-1} x$	$-\infty \le x \le \infty$	$0 < y < \pi$
$\text{cosec}^{-1} x$	$\lvert x \rvert \ge 1$	$-\frac{\pi}{2} < y < 0 \cup 0 < y < \frac{\pi}{2}$

Derivatives of Inverse Trigonometric Functions

Standard Derivatives:

1. $$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

2. $$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

3. $$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

4. $$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

5. $$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

6. $$\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

3.2.1 Derivatives of $y = \sin^{-1} x$. *(proof)*

Recall: $y = \sin^{-1} x \Leftrightarrow x = \sin y$

for $x \in [-1, 1]$ and $y \in [-\pi/2, \pi/2]$.

Because the sine function is differentiable on $[-\pi/2, \pi/2]$, the inverse function is also differentiable.

To find its derivative we proceed implicitly:

Given $\sin y = x$. Differentiating w.r.t. x:

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$

$$\cos y \frac{dy}{dx} = 1$$

$$\therefore \qquad \frac{dy}{dx} = \frac{1}{\cos y}$$

Since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $\cos y \ge 0$, hence

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$$

Example 3.3:

1. Differentiate each of the following functions.

(a) $f(x) = \tan^{-1}\sqrt{x}$

(b) $g(t) = \sin^{-1}(1-t)$

(c) $h(x) = \sec^{-1} e^{2x}$

2. Find the derivative of:

(a) $y = (\tan^{-1} x^2)^4$

(b) $f(x) = \ln(\sin^{-1} 4x)$

3. Find the derivative of $y = \tan^{-1}(\tan(3t^2 - 1))$.

4. Find the derivative $\dfrac{dy}{dx}$ if

(a) $x\tan^{-1} y = x^2 + y$

(b) $\sin^{-1}(xy) + \dfrac{\pi}{2} = \cos^{-1} y$

Summary

If u is a differentiable function of x, then

1. $\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$

2. $\frac{d}{dx}(\cos^{-1} u) = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}$

3. $\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2} \frac{du}{dx}$

4. $\frac{d}{dx}(\cot^{-1} u) = \frac{-1}{1+u^2} \frac{du}{dx}$

5. $\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dx}$

6. $\frac{d}{dx}(\csc^{-1} u) = \frac{-1}{|u|\sqrt{u^2-1}} \frac{du}{dx}$

3. 3 Derivatives of Inverse hyperbolic Functions

Recall: **Inverse Hyperbolic Functions**

Function	Domain	Range
$y=\sinh^{-1}x$	$(-\infty,\infty)$	$(-\infty,\infty)$
$y=\cosh^{-1}x$	$[1,\infty)$	$[0,\infty)$
$y=\tanh^{-1}x$	$(-1, 1)$	$(-\infty,\infty)$
$y=\coth^{-1}x$	$(-\infty,-1)\cup(1,\infty)$	$(-\infty,0)\cup(0,\infty)$
$y=\text{sech}^{-1}x$	$(0, 1]$	$[0,\infty)$
$y=\text{cosech}^{-1}x$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$

Function	Logarithmic form
$y=\sinh^{-1}x$	$\ln\left(x+\sqrt{x^2+1}\right)$
$y=\cosh^{-1}x$	$\ln\left(x+\sqrt{x^2-1}\right)$
$y=\tanh^{-1}x$	$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right);\ \lvert x\rvert<1$

3.3.1 Prove that $\dfrac{d}{dx}(\sinh^{-1} x) = \dfrac{1}{\sqrt{1+x^2}}$

$$\textit{Recall:}\ \ y = \sinh^{-1} x \Leftrightarrow x = \sinh y$$

To find its derivative we proceed implicitly:

- Given $x = \sinh y$. Differentiating w.r.t. x:

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sinh y)$$

$$1 = \cosh y \frac{dy}{dx}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\cosh y}$$

- Since $-\infty < y < \infty$, $\cosh y \geq 0$, so using the identity $\cosh^2 y - \sinh^2 y = 1$:

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$$

$$\therefore \frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$$

- Other ways to obtain the derivatives are:

(a) $y = \sinh^{-1} x \Leftrightarrow x = \sinh y$ then

$x = \dfrac{e^{y} - e^{-y}}{2}$. Hence, find $\dfrac{dy}{dx}$.

(b) $y = \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$.

Hence, find $\dfrac{dy}{dx}$.

Standard Derivatives of Inverse Hyperbolic Functions

Function, y	**Derivatives, $\frac{dy}{dx}$**
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}; \ \lvert x\rvert > 1$
$\tanh^{-1} x$	$\frac{1}{1-x^2}; \ \lvert x\rvert < 1$
$\coth^{-1} x$	$\frac{1}{1-x^2}; \ \lvert x\rvert > 1$
$\text{sech}^{-1} x$	$-\frac{1}{x\sqrt{1-x^2}}; \ 0 < x < 1$
$\text{cosech}^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{1+x^2}}; \ x \neq 0$

Generalized Form

$y = f(u);$ $u = g(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$
$\sinh^{-1} u$	$\dfrac{1}{\sqrt{u^2+1}} \dfrac{du}{dx}$
$\cosh^{-1} u$	$\dfrac{1}{\sqrt{u^2-1}} \dfrac{du}{dx}; \ \lvert u \rvert > 1$
$\tanh^{-1} u$	$\dfrac{1}{1-u^2} \dfrac{du}{dx}; \ \lvert u \rvert < 1$
$\coth^{-1} u$	$\dfrac{1}{1-u^2} \dfrac{du}{dx}; \ \lvert u \rvert > 1$
$\text{sech}^{-1} u$	$-\dfrac{1}{u\sqrt{1-u^2}} \dfrac{du}{dx}; \ 0 < u < 1$
$\text{cosech}^{-1} u$	$\dfrac{1}{\lvert u \rvert \sqrt{1+u^2}} \dfrac{du}{dx}; \ u \neq 0$

Example 3.4: Find the derivatives of

(a) $y = \sinh^{-1}(1-3x)$

(b) $y = \cosh^{-1}\left(\frac{1}{x}\right)$

(c) $y = e^x \operatorname{sech}^{-1} x$

(d) $y = \sinh^{-1}(\tan 3x)$

(e) $f(t) = \dfrac{\tanh^{-1} t^2}{1-\sec t}$

(f) $y = \sqrt{\coth^{-1} x}$

(g) $y = \cos\ 4x\ \cosh^{-1} 4x$

(h) $y^3 - \sinh^{-1} xy = 0$

3.4 References

1. George B. Thomas, Maurice D. Weir, Joel R. Hass, and Frank R. Giordano. 2005. *Thomas' Calculus Early Transcendental (11th Edition) (Thomas Series)*. Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA.

2. Abdul Wahid Md Raji, Hamisan Rahmat, Ismail Kamis, Mohd Nor Mohamad, Ong Chee Tiong. ***Engineering mathematics I***, Penerbit UTM, 2012.