SSCM 1023 MATHEMATICAL METHODS I

POLAR COORDINATES

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POLAR COORDINATES

1.1 Parametric Equations

1.1.1 Definition:

Equations $x = f(t),\ y = g(t)$ that express *x* and *y* in *t* is known as **parametric equations**, and *t* is called the **parameter**.

How the parameter may be eliminated from the parametric equations to obtain the Cartesian equations?

- no specific method
- use algebraic manipulation

Example 1: Form Cartesian equations by eliminating parameter *t* in the following equations:

(a) $x = 2t,\ y = 4t^2 - 1$

(b) $x = 4\sin t,\ y = 2\cos^2 t$

(c) $x = e^t,\ y = e^{-t}$

(d) $x = t^3,\ y = 3\ln t$

1.1.2 Curve Sketching of Parametric Equations

2 ways:
- Constructing tables of x and y for specific values of t.
- Eliminating the parameter.

Example 2:

Sketch the graph of the following equations

(a) $x = 2t,\ y = 4t^2 - 1$

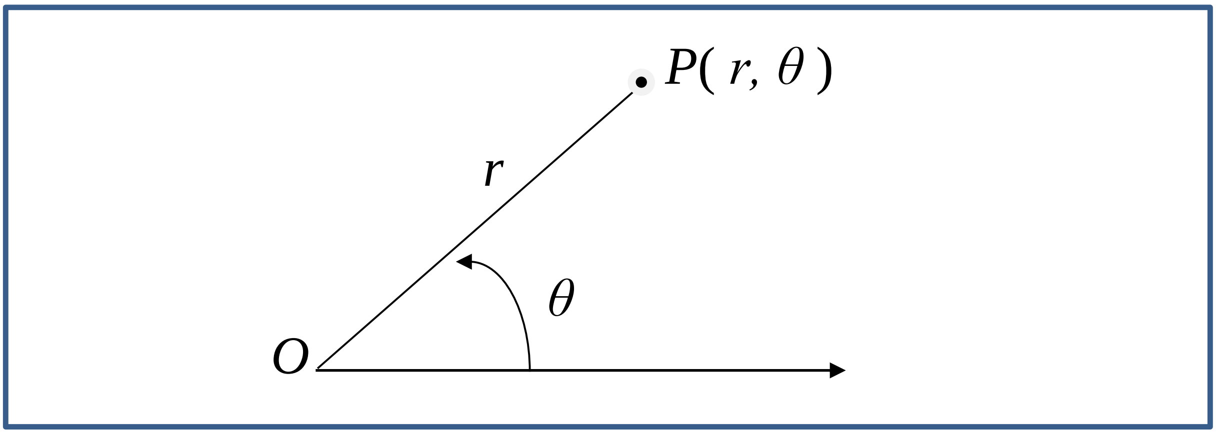
(b) $x = 3t - 5,\ y = 2t + 5$

1.2 Polar Coordinates System

Definition:

The polar coordinates of point P is written as an ordered pair (r,θ), that is $P(r,\theta)$ where

r - distance from origin to P
θ - angle from polar axis to the line OP



Note:

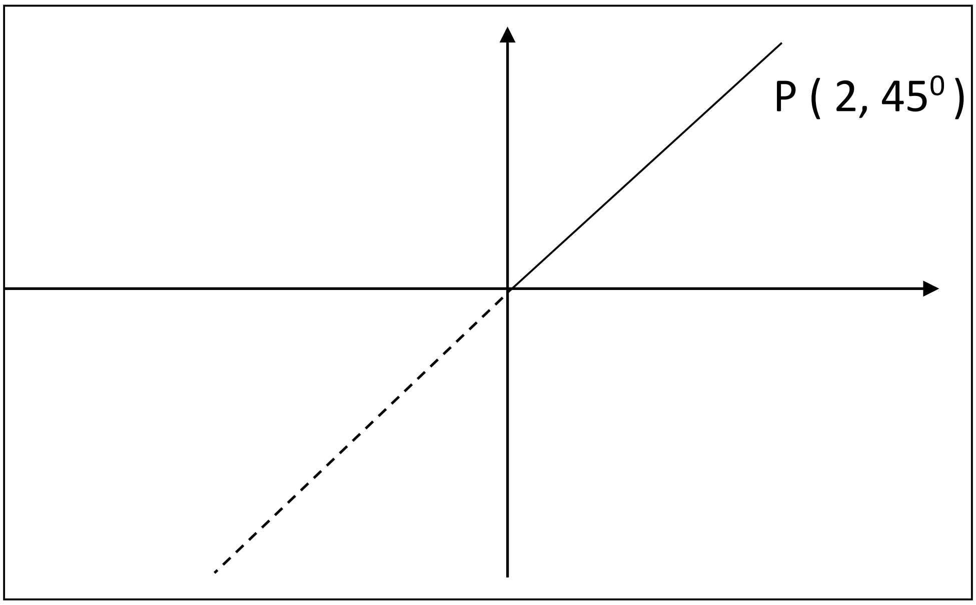
(i) θ is positive in anticlockwise direction, and it is negative in clockwise direction.
(ii) Polar coordinate of a point is not unique.
(iii) A point $(-r,\theta)$ is in the opposite direction of point (r,θ).

Example 3: Plot the following set of points in the same diagram:

(a) $(3, 225^\circ)$, $(1, 225^\circ)$, $(-3, 225^\circ)$

(b) $\left(2, \frac{\pi}{3}\right)$, $\left(2, -\frac{\pi}{3}\right)$, $\left(-2, \frac{\pi}{3}\right)$

For every point $P(r, \theta)$ in $0 \leq \theta \leq 2\pi$, there exist 3 more coordinates that represent the point P.

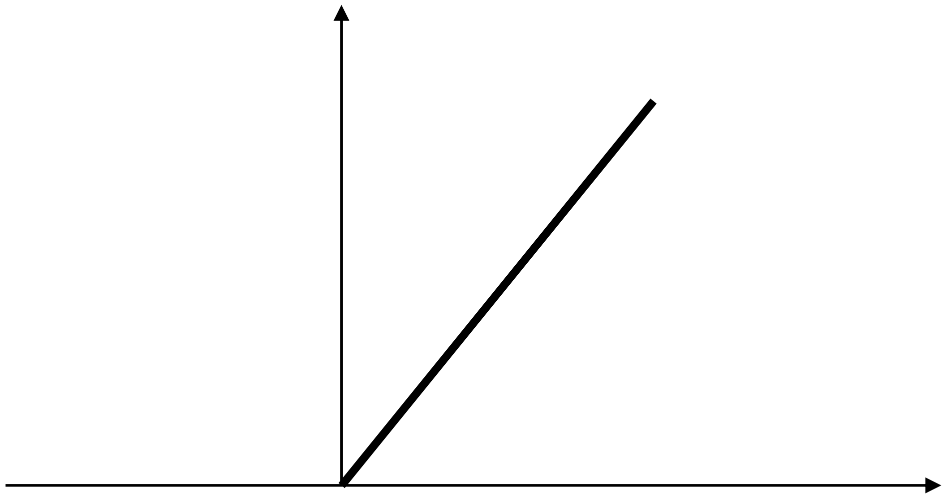


Example 4:

Find all possible polar coordinates of the points whose polar coordinates are given as the following:

(a) $P(1, 45^\circ)$ (b) $Q(2, -60^\circ)$ (c) $R(-1, 225^\circ)$

1.3 Relationship between Cartesian and Polar Coordinates



1) Polar ⟹ Cartesian

$$x = r\cos\theta \qquad y = r\sin\theta$$

2) Cartesian ⟹ Polar

$$r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x}$$

Example 5: Find the Cartesian coordinates of the points whose polar coordinates are given as

(a) $\left(1, \frac{7\pi}{4}\right)$ (b) $\left(-4, \frac{2\pi}{3}\right)$ (c) $\left(2, -30^\circ\right)$

Example 6: Find all polar coordinates of the points whose rectangular coordinates are given as

(a) $(11,5)$ (b) $(0,2)$ (c) $(-4,-4)$

1.4 Forming polar equations from Cartesian equations and vice-versa.

To change the equation in Cartesian coordinates to polar coordinates, and conversely, use equation

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2 + y^2}$$

Example 7: Express the following rectangular equations in polar equations.

(a) $y = x^2$ (b) $x^2 + y^2 = 16$ (c) $xy = 1$

Example 8*:* Express the following polar equations in rectangular equations and sketch the graph.

(a) $r = 2\sin\theta$ (b) $r = \dfrac{3}{4\cos\theta + 5\sin\theta}$

(c) $r = 4\cos\theta + 4\sin\theta$ (d) $r = \tan\theta\sec\theta$

e) $r^2 = \dfrac{2}{3\cos^2\theta - 1}$

1.5 Graph Sketching of Polar Equations

There are two methods to sketch a graph of $r = f(\theta)$

(1) Form a table for r and θ. $(0 \le \theta \le 2\pi)$.

From the table, plot the (r, θ) points.

(2) Symmetry test of the polar equation.

The polar equations is symmetrical about:

(a) x-axis if $(r, -\theta) = f(\theta)$ or $(-r, \pi - \theta) = f(\theta)$.

- consider θ in range [0, 180^0] only.

(b) y-axis if $(r, \pi - \theta) = f(\theta)$ or $(-r, -\theta) = f(\theta)$.

- consider θ in range [0, 90^0] **and** [270^0, 360^0]

(c) origin if $(r, \pi + \theta) = f(\theta)$ or $(-r, \theta) = f(\theta)$.

- consider θ in range [0, 180^0] **or** [180^0, 360^0]

* if symmetry at all, consider θ in range [0, 90^0]

only.

Example 9: Sketch the graph of $r = 2\sin\theta$

Solution: (Method 1)

Here is the complete table

θ	0	30	60	90	120	150	180	210
$r = 2\sin\theta$	0	1.0	1.732	2	1.732	1	0	-1.0

θ	240	270	300	330	360
$r = 2\sin\theta$	-1.732	-2	-1.732	-1	0

Then, plot the points on the diagram:

Method 2:

Symmetrical test for $f(\theta) = 2\sin\theta$

Symmetry	Symmetrical test
About x-axis	
About y-axis	
About origin	

Since r is symmetry at y-axis, then consider θ in the range $[0, 90^0]$ **and** $[270^0, 360^0]$

θ	0	30	60	90	270	300	330	360
$r = 2\sin\theta$	0	1.0	1.732	2	-2	-1.732	-1	0

Then, plot the points on the diagram:

Example 10: Sketch the graph of $r = \frac{3}{2} - \cos\theta$

Symmetry	Symmetrical test
About x-axis	
About y-axis	
About origin	

Since r symmetry at x-axis, consider θ in range

$[0, 180^0]$ only.

θ	0	30	60	90	120	150	180
$r = \frac{3}{2} - \cos\theta$							

Then, plot the points on the diagram:

Example 11: Sketch the graph of $r = 2\sin^2\theta$

Symmetry	Symmetrical test
About x-axis	
About y-axis	
About origin	

Since r symmetry at _______________, consider θ in range _______________ only.

θ							
$r = 2\sin^2\theta$							

Then, plot the points on the diagram:

1.6 Finding the Intersection Points between Two Curves In Polar Coordinates

Steps:

1. Solve simultaneous equations between 2 curves and determine the intersection points.
   -if one of the curves is a line (i.e. $\theta = k$), we need to find intersection point for $\theta = k - \pi$.

2. Check whether the curves intersect at the origin.
   - Test for $r = 0$. If θ exist, it means the 2 curves intersect at the origin.

Example 12:

Find the points of intersection of the circle $r = 2\cos\theta$ and $r = 2\sin\theta$ for $0 \le \theta \le \pi$

Example 13:

Find the points of intersection of the curves $r = \frac{3}{2} - \cos\theta$ and $\theta = \frac{2\pi}{3}$.

Example 14:

A polar equation is given as $r = 2 - 5\sin\theta$.

a) Show that the curve is symmetrical about the *y*-axis and passes through the origin.

b) Make a suitable graph for $-90^0 \le \theta \le 90^0$. Use the table and the information in part a) to make a full sketch of the graph.

c) Find the intersection points of the graph and the straight line $\theta = \dfrac{11\pi}{12}$

1.7 References

1. George B. Thomas, Maurice D. Weir, Joel R. Hass, and Frank R. Giordano. 2005. *Thomas' Calculus Early Transcendental (11th Edition) (Thomas Series)*. Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA.

2. Abdul Wahid Md Raji, Hamisan Rahmat, Ismail Kamis, Mohd Nor Mohamad, Ong Chee Tiong. ***Engineering mathematics I***, Penerbit UTM, 2012.