Chap 4: Exponential Smoothing

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Chap 4: Exponential Smoothing

Outline:

- Introduction to exponential smoothing
- Simple exponential smoothing method
- Holt's exponential smoothing
- Winter's exponential smoothing
- Multiplicative Holt-winter's model
- Additive Holt-winter's model

Introduction to exponential smoothing

Exponential smoothing methods is a method for continually revising an estimate or forecast by accounting for fluctuations in the data. These method can be categories into 3 methods

- The simple exponential smoothing method is the single smoothing (SES) method with only one parameter and allows forecasting for series with stationary or no trend.
- Holt's method makes use of two different parameters and allows forecasting for series with trend.
- Holt-Winters' method involves three smoothing parameters to smooth the data, the trend, and the seasonal index.

Simple exponential smoothing method

The Simple Exponential Smoothing method used for time series have no trend (stationary) and the mean (or level) of the time series y_t is slowly changing over time.

Formally, the simple exponential smoothing model is

$$F_{t+1} = \alpha y_t + (1-\alpha) F_t$$

F_{t+1} = forecast for the next period.

α = smoothing constant.

y_t = observed value of series in period *t*.

F_t = old forecast for period *t*.

Procedures of simple exponential smoothing method

Step 1: Use the mean of the series as the initial F_t at time period $t = 0$

$$F_1 = \bar{y} = \sum_{t=1}^{n} y_t / n$$

Step 2: Calculate the updated estimate by using formula

$$F_{t+1} = \alpha y_t + (1-\alpha) F_t$$

where α is a smoothing constant between 0 and 1.

Step 3: Calculate the value of F_{t+1} for α = 0.01, 0.02, 0.03, …, 0.99 and the sum of squared forecast error (SSE) is computed for each, where

$$SSE = \sum (y_t - F_t)^2$$

Step 4: Determination of α *is usually judgmental and subjective* and often based on trial-and -error experimentation. The value of α with the smallest SSE is chosen for use in producing the future forecasts.

Step 5: *1*-step-ahead forecast made at time *T*

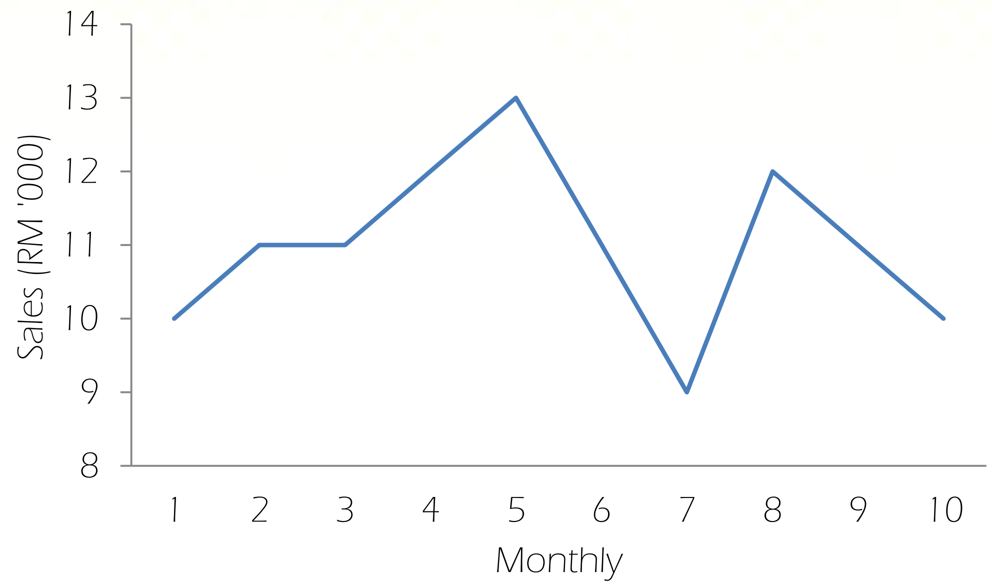
$$F_{t+1} = \alpha\, y_t + (1-\alpha) F_t$$

Example: Simple exponential smoothing

Yearly sales data for certain product were collected over 10-month period, as given below.

Sales (In RM '000)

Month	**Year 1**
January	10
February	11
March	11
April	12
May	13
June	11
July	9
August	12
September	11
October	10



The plot of these data suggests that there is no trend or seasonal pattern.

Solution: Simple exponential smoothing

Compute F_1 by averaging the time series values.

$$F_1 = \bar{y} = \sum_{t=1}^{n} y_t / n = 9.4$$

Let $\alpha = 0.2$, the first period forecast is

$$F_2 = 0.2(y_1) + (1 - 0.2)F_1 = 0.2(10) + 0.8(9.4) = 9.52$$

The second period forecast is

$$F_3 = 0.2(y_2) + (1 - 0.2)F_2 = 0.2(11) + 0.8(9.4) = 9.2160$$

By continuing this process, we can generate the forecasts for the 4th through the 10th time periods as shown in Table 1.

Solution: Continue

Table 1: Forecasting Values Using Single Exponential Smoothing with α=0.2

t	y(t)	F(t)	Error	Error^2
1	10	9.4000	0.6000	0.3600
2	8	9.5200	-1.5200	2.3104
3	10	9.2160	0.7840	0.6147
4	4	9.3728	-5.3728	28.8670
5	12	8.2982	3.7018	13.7030
6	11	9.0386	1.9614	3.8471
7	6	9.4309	-3.4309	11.7709
8	12	8.7447	3.2553	10.5970
9	11	9.3958	1.6042	2.5736
10	10	9.7166	0.2834	0.0803
			SSE	74.7240

Results associated with different values of α

α	SSE
0.1	68.5371
0.2	74.7240
0.3	81.5214
0.4	89.3826
0.5	98.6005
0.6	109.3832
0.7	121.9189
0.8	136.4143
0.9	153.1202

Holt's Exponential smoothing

- Holt's two parameter exponential smoothing method is an extension of simple exponential smoothing.
- It adds a growth factor (or trend factor) to the smoothing equation as a way of adjusting for the trend.
- It model suitable when a time series is increasing or decreasing approximately at a fixed rate.

Holt's Exponential Smoothing

Holt's smoothing approach employs two smoothing constants, denoted by α and γ. The Holt's smoothing model is

$$F_{t+m} = L_t + mb_t$$

where

$$L_t = \alpha y_t + (1-\alpha)(L_{t-1} + b_{t-1}) \qquad \text{(level series)}$$

$$b_t = \beta (L_t - L_{t-1}) + (1-\beta) b_{t-1} \qquad \text{(trend estimate)}$$

α = smoothing constant for the level (0 ≤ α ≤ 1)
γ = smoothing constant for the trend (0 ≤ γ ≤ 1)
m = periods to be forecast into the future

Procedures of Holt's exponential smoothing

Step 1: Obtain initial estimates L_1 and b_1 by set $L_1 = y_1$ and

$$b_1 = y_2 - y_1 \quad \text{or} \quad b_1 = \frac{y_4 - y_1}{3} \quad \text{or} \quad b_1 = 0$$

Step 2: Calculate a point forecast of y_2 from time $t = 1$

$$F_2 = L_1 + b_1$$

Step 3: Update the estimates L_t and b_t by using

$$L_t = \alpha\, y_t + (1-\alpha)(L_{t-1} + b_{t-1})$$

and

$$b_t = \beta(L_t - L_{t-1}) + (1-\beta)b_{t-1}$$

with α and β are set 0.01, 0.02, …, 0.99.

Step 4: Find the best combination of α and γ that minimizes SSE (or MSE).

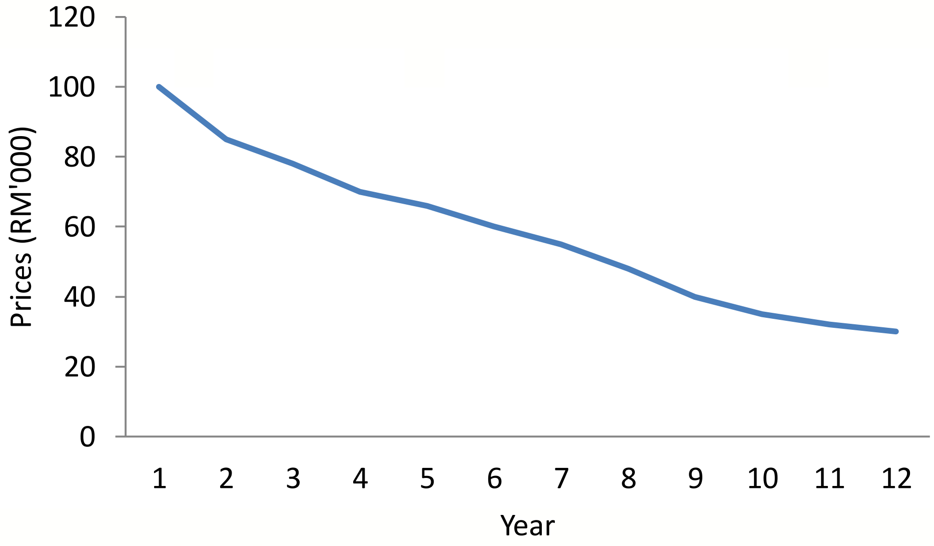
Step 5: m-step-ahead forecast made at time T

$$F_{t+m} = L_t + mb_t$$

Example: Holt's exponential smoothing

Use the example of the Price of certain car (RM'000) from 2000 to 2011 as an illustration

Year	t	Prices
2000	1	100
2001	2	85
2002	3	78
2003	4	70
2004	5	66
2005	6	60
2006	7	55
2007	8	48
2008	9	40
2009	10	35
2010	11	32
2011	12	30



Overall an downward trend and no seasonal pattern

Solution: Holt's exponential smoothing

Set $L_0 = y_1$ and $b_0 = 0$. The first period forecast is

$$F_1 = L_0 + b_0 = 100$$

Continuing the process for period 2 . Let $\alpha = 0.2$ and $\beta = 0.4$

$$L_1 = 0.2\,y_1 + (1-0.2)(L_0 + b_0) = 0.2(100) + 0.8(100) = 100$$

$$b_1 = 0.4\,(L_1 - L_0) + (1-0.4)b_0 = 0.4(100-100) + 0.6(0) = 0$$

$$F_2 = L_1 + b_1 = 100$$

The remaining forecasts in Table 2 were calculated in the same manner.

Solution:

Table 2: Forecasting Values Using Holt's Exponential Smoothing with $\alpha = 0.2$ and $\beta = 0.4$

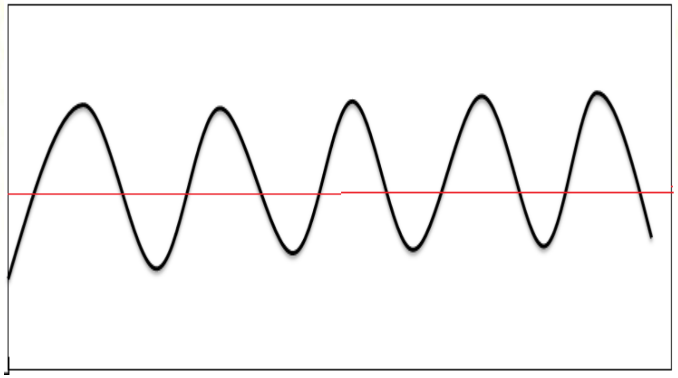
Year	t	Actual	L(t)	b(t)	F(t)	Error	$Error^2$
1999	0		100.000	0.000			
2000	1	100	100.000	0.000	100.000	0.000	0.000
2001	2	85	97.000	-1.200	100.000	-15.000	225.000
2002	3	78	92.240	-2.624	95.800	-17.800	316.840
2003	4	70	85.693	-4.193	89.616	-19.616	384.787
2004	5	66	78.400	-5.433	81.500	-15.500	240.235
2005	6	60	70.373	-6.471	72.966	-12.966	168.127
2006	7	55	62.122	-7.183	63.903	-8.903	79.255
2007	8	48	53.551	-7.738	54.939	-6.939	48.154
2008	9	40	44.651	-8.203	45.814	-5.814	33.797
2009	10	35	36.158	-8.319	36.448	-1.448	2.096
2010	11	32	28.672	-7.986	27.839	4.161	17.310
2011	12	30	22.548	-7.241	20.686	9.314	86.758
		Forecast value in 2012			15.308	SSE	1602.360

By using Solver in Excel , the value of $\alpha = 1$ and $\beta = 0.4543$ were found is the best that minimizes SSE = 265.066.

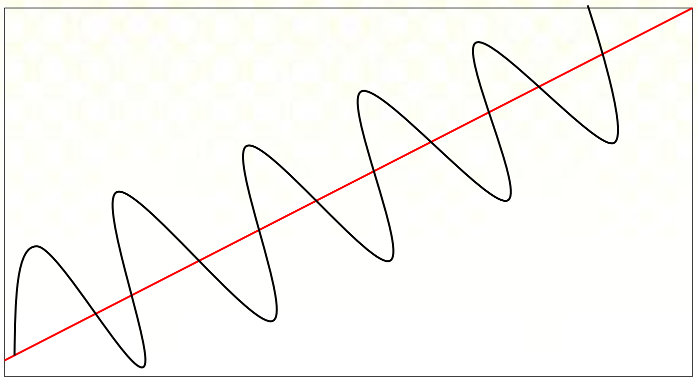
Holt-winter's exponential smoothing

- Winter's exponential smoothing model is the second extension of the basic Exponential smoothing model.
- It is used for data that exhibit both trend and seasonality.
- It is a three parameter model that is an extension of Holt's method.
- An additional equation adjusts the model for the seasonal component.

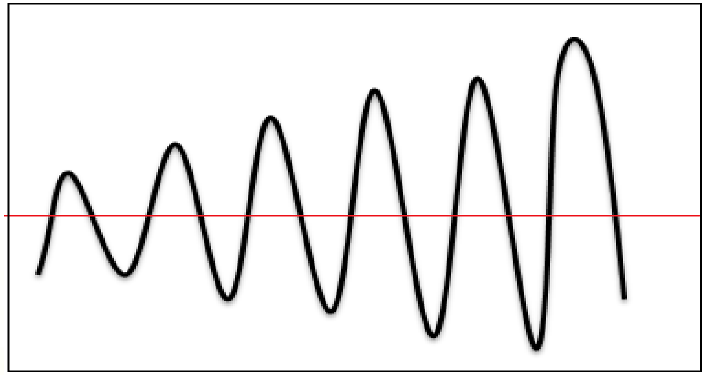
Holt-winter’s exponential smoothing



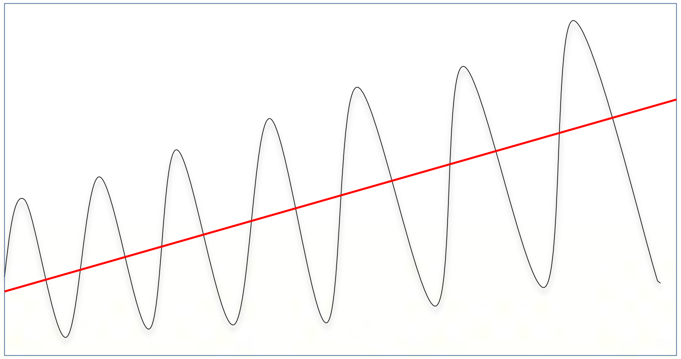
1. No trend and additive seasonal variability



2. Additive seasonal variability with an additive trend



3. Multiplicative seasonal variability with no trend



4. Multiplicative seasonal variability with a multiplicative trend

Multiplicative Holt-winter's model

- Multiplicative Holt-Winter's model is

$$F_{t+m} = (L_t + mb_t)S_{t+m-s}$$

where

$$L_t = \alpha \frac{y_t}{S_{t-s}} + (1-\alpha)(L_{t-1} + b_{t-1}) \quad \text{(level series)}$$

$$b_t = \beta(L_t - L_{t-1}) + (1-\beta)b_{t-1} \quad \text{(trend estimate)}$$

$$S_t = \gamma \frac{y_t}{L_t} + (1-\gamma)S_{t-s} \quad \text{(seasonality factor)}$$

where α, β and γ are smoothing constants between 0 and 1, s = number of seasons in a year (s = 12 for monthly data, and s = 4 for quarterly data)

Procedures of Multiplicative Holt-Winters Method

Step 1: Obtain initial values for the level L_0, the growth rate b_0, and the seasonal factors $S_1, S_2, S_3, \ldots, S_s$, by set

Initialize level as:

$$L_s = \frac{1}{s}(y_1 + y_2 + \cdots y_s)$$

Initialize trend as

$$b_s = \frac{1}{s}\left(\frac{y_{s+1} - y_1}{s} + \frac{y_{s+2} - y_2}{s} + \cdots + \frac{y_{s+s} - y_s}{s}\right)$$

Initialize seasonal indices as

$$S_1 = \frac{y_1}{L_s}, S_2 = \frac{y_2}{L_s}, \cdots, S_s = \frac{y_s}{L_s}$$

Step 2: Calculate a point forecast of F_{t+1} using the initial values

$$F_{t+1} = (L_t + b_t)S_{t+1-s}$$

Step 3: Update the estimates L_t, b_t, and S_t by using the formula

$$L_t = \alpha \frac{y_t}{S_{t-s}} + (1-\alpha)(L_{t-1} + b_{t-1})$$

$$b_t = \beta(L_t - L_{t-1}) + (1-\beta)b_{t-1}$$

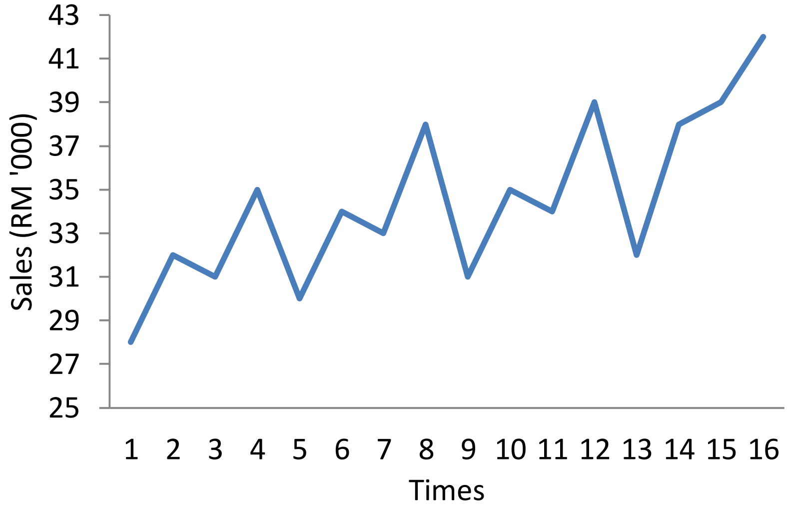
$$S_t = \gamma \frac{y_t}{L_t} + (1-\gamma)S_{t-s}$$

Step 4: Find the most suitable combination of α, β and γ that minimizes SSE (or MSE)

Example: Multiplicative Holt-Winters Method

- Use the quarterly retail sales data (RM Million) example as an illustration

Year	Quarter	t	Sales
2005	1	1	28
	2	2	32
	3	3	31
	4	4	35
2006	1	5	30
	2	6	34
	3	7	33
	4	8	38
2007	1	9	31
	2	10	35
	3	11	34
	4	12	39
2008	1	13	32
	2	14	38
	3	15	39
	4	16	42



The series shows linear upward trend and the magnitude of the seasonal span increases as the level of the time series increases

Solution: Multiplicative Holt-Winters Method

Table 3 presents an example of the Winter's multiplicative smoothing procedure used for $\alpha = 0.4, \beta = 0.1$ and $\gamma = 0.3$.

Year	Quarter	t	y(t)	L(t)	b(t)	S(t)	F(t)	Error	Error^2
2005	1	1	28			0.889			
	2	2	32			1.016			
	3	3	31			0.984			
	4	4	35	31.500	0.563	1.111			
2006	1	5	30	32.738	0.630	0.897	28.500	1.500	2.250
	2	6	34	33.408	0.634	1.016	33.897	0.103	0.011
	3	7	33	33.838	0.614	0.981	33.502	-0.502	0.252
	4	8	38	34.351	0.604	1.110	38.280	-0.280	0.078
2007	1	9	31	34.795	0.588	0.895	31.359	-0.359	0.129
	2	10	35	35.003	0.550	1.011	35.963	-0.963	0.928
	3	11	34	35.189	0.513	0.977	34.893	-0.893	0.798
	4	12	39	35.480	0.491	1.107	39.616	-0.616	0.380
2008	1	13	32	35.880	0.482	0.894	32.204	-0.204	0.042
	2	14	38	36.845	0.530	1.017	36.779	1.221	1.491
	3	15	39	38.394	0.632	0.989	36.511	2.489	6.195
	4	16	42	38.598	0.589	1.101	43.183	-1.183	1.400
								SSE	13.954

By using Solver in Excel , the value of $\alpha = \beta = \gamma = 0$ were found is the best that minimizes SSE = 9.782

Additive Holt-winter's method

The Holt's Winters' additive method is

$$F_{t+m} = L_t + b_t m + S_{t+m-s}$$

where

$$L_t = \alpha(y_t - S_{t-s}) + (1-\alpha)(L_{t-1} + b_{t-1})$$

$$b_t = \beta(L_t - L_{t-1}) + (1-\beta)b_{t-1}$$

$$S_t = \gamma(y_t - L_t) + (1-\gamma)S_{t-s}$$

Where α, β and γ are smoothing constants between 0 and 1, s = number of seasons in a year (s = 12 for monthly data, and s = 4 for quarterly data)

Procedures of additive Holt-Winters method

Step 1: Obtain initial values for the level L_0, the growth rate b_0, and the seasonal factors $S_1, S_2, S_3, \ldots, S_s$, by set

Initialize level as:

$$L_s = \frac{1}{s}(y_1 + y_2 + \cdots y_s)$$

Initialize trend as

$$b_s = \frac{1}{s}\left(\frac{y_{s+1} - y_1}{s} + \frac{y_{s+2} - y_2}{s} + \cdots + \frac{y_{s+s} - y_s}{s}\right)$$

Initialize seasonal indices as

$$S_1 = y_1 - L_s, \quad S_2 = y_2 - L_s, \cdots, S_s = Y_s - L_s$$

Step 2: Calculate a point forecast of F_{t+1} using the initial values

$$F_{t+m} = L_t + b_t m + S_{t+m-s}$$

Step 3: Update the estimates L_t, b_t, and S_t by using the formula

$$L_t = \alpha(y_t - S_{t-s}) + (1-\alpha)(L_{t-1} + b_{t-1})$$

$$b_t = \beta(L_t - L_{t-1}) + (1-\beta)b_{t-1}$$

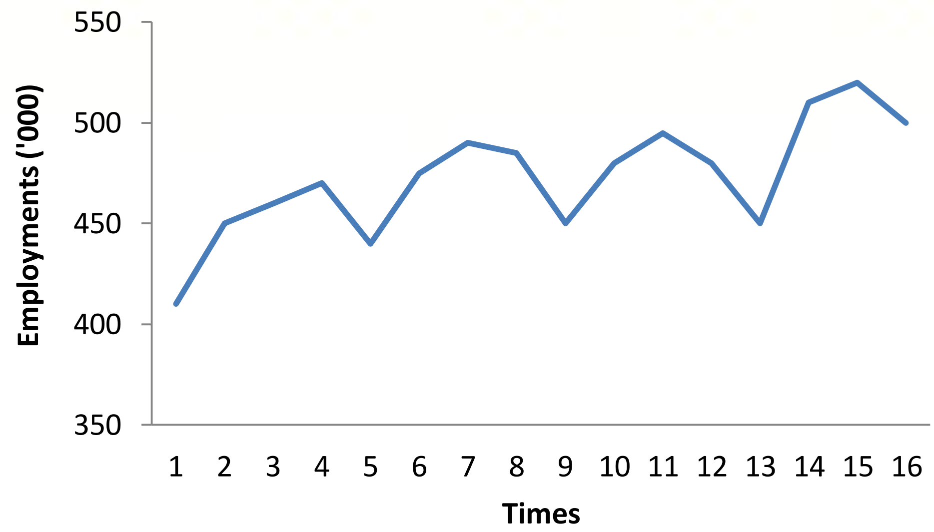
$$S_t = \gamma(y_t - L_t) + (1-\gamma)S_{t-s}$$

Step 4: Find the most suitable combination of α, β and γ that minimizes SSE (or MSE)

Example: Additive Holt-Winters Method

- Consider the construction employment ('000) example,

t	Employment
1	410
2	450
3	460
4	470
5	440
6	475
7	490
8	485
9	450
10	480
11	495
12	480
13	450
14	510
15	520
16	500



The series shows linear upward trend and the magnitude of seasonal span is almost constant as the level of the time series increases

Solution: Additive Holt-Winters Method

Table 4 presents an example of the Winter's additive smoothing procedure used for $\alpha = 0.4, \beta = 0.1$ and $\gamma = 0.3.$

t	y(t)	L(t)	b(t)	S(t)	F(t)	Error	Error^2
1	410			-37.500			
2	450			2.500			
3	460			12.500			
4	470	447.500	6.250	22.500			
5	440	463.250	7.200	-33.225	416.250	23.750	564.063
6	475	471.270	7.282	2.869	472.950	2.050	4.203
7	490	478.131	7.240	12.311	491.052	-1.052	1.107
8	485	476.223	6.325	18.383	507.871	-22.871	523.088
9	450	482.819	6.352	-33.103	449.323	0.677	0.459
10	480	484.355	5.871	0.702	492.040	-12.040	144.957
11	495	487.211	5.569	10.954	502.536	-7.536	56.793
12	480	480.315	4.323	12.774	511.163	-31.163	971.154
13	450	484.024	4.261	-33.379	451.534	-1.534	2.354
14	510	496.690	5.102	4.484	488.987	21.013	441.557
15	520	504.694	5.392	12.260	512.746	7.254	52.619
16	500	500.942	4.478	8.659	522.859	-22.859	522.544
						SSE	3284.897

By using Solver in Excel, the value of $\alpha = 0.3173$, $\beta = 0$ and $\gamma = 0.5311$ were found is the best that minimizes SSE = 3053.525.