SSCE1993 ENGINEERING MATHEMATICS

VECTOR FUNCTIONS

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Vector Functions

Definitions: Let D be a subset of real numbers. A vector function $\mathbf{F}$ with domain D is a correspondence that associates each number t in D with a unique vector $\mathbf{F}(t)$ in two or three dimensions. The range of $\mathbf{F}$ consists of all vectors $\mathbf{F}(t)$ of t in D.

A vector function whose range in R^2 and R^3 are

$$\mathbf{F}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$$
$$\mathbf{F}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

where f, g and h are scalar valued functions of real number t defined on domain of D respectively.

Vector Derivatives

- Let $\mathbf{F}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ for some differentiable scalar function $f(t), g(t)$and $h(t)$. Then

$$\mathbf{F}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

- Let $\mathbf{F}(t)$ be differentiable at t_0 and that $\mathbf{F}'(t_0) \neq 0$. Then the tangent vector to the graph $\mathbf{F}(t)$ at point $t = 0$ is given as $\mathbf{F}'(t)$.

- The unit tangent vector on the graph of $\mathbf{F}(t)$, denoted by $\mathbf{T}$, is defined as $\mathbf{T}(t) = \frac{\mathbf{F}'(t)}{\|\mathbf{F}'(t)\|}$

- If $\mathbf{T}'(t) \neq 0$, the principle unit normal vector to the graph $\mathbf{F}(t)$, denoted by $\mathbf{N}$, is defined as $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$

Vector Integrals

Let $\mathbf{F}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ where f, g and h are continuous is t. Then the integral of $\mathbf{F}(t)$ is the vector function

$$\int \mathbf{F}(t)\,dt = \left[\int f(t)dt\right]\mathbf{i} + \left[\int g(t)dt\right]\mathbf{j} + \left[\int g(t)dt\right]\mathbf{k}$$

Class Activity

Find the unit tangent vector $\mathbf{T}$ and the principle unit normal vector $\mathbf{N}$ of $\mathbf{F}(t)$ at the given t

1. $\mathbf{F}(t) = t\mathbf{i} + t\mathbf{j} + \mathbf{k}, \quad t = 2$
2. $\mathbf{F}(t) = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + 3t\,\mathbf{k}$, at $t = \pi$

Motion in space

If a particle moves along a curve given by the position vector $\mathbf{r}(t)$, where t is time, then

$$\mathbf{v} = \frac{d\mathbf{v}}{dt}$$

is the velocity and $||\mathbf{v}||$ is the speed of the particle respectively, while the acceleration of the particle is given by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$$

Class Activity

1. The position vector of a moving particle after a time t is given by
$$\mathbf{r}(t) = e^{-t}\mathbf{i} + 2\cos 3t\ \mathbf{j} + 2\sin 3t\ \mathbf{k}$$

Find the velocity , speed and the acceleration of the particle at $t = 0$.

2. The velocity of a particle moving in space is

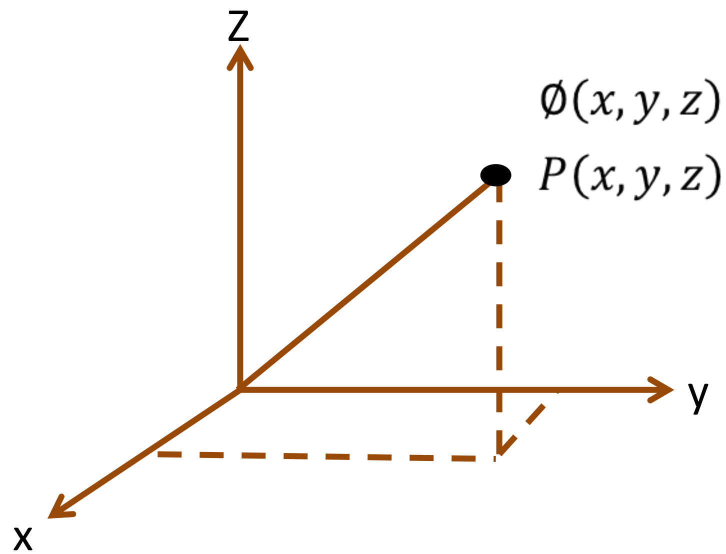
$$\mathbf{v}(t) = t^2\,\mathbf{i} + (6t+1)\,\mathbf{j} + 8t^3\ \mathbf{k}$$

Find the particle's position as a function of t if
$$\mathbf{r}(0) = \mathbf{2i} + \mathbf{3\,j} + \mathbf{k}$$

Scalar Fields

Definition:

If to each point $P(x, y, z)$ of a region in space there is made to correspond to a scalar quantity $\emptyset(x, y, z)$, then $\emptyset(x, y, z)$ is a scalar function and we say there is a scalar field in the region.



Scalar Fields

Definition:

Let $\emptyset(x, y, z)$ be a scalar function that is defined and differentiable in a region. Then the gradient of $\emptyset$, written as $\mathbf{grad}\ \emptyset$, is defined as

$$\mathbf{grad}\ \emptyset = \nabla\emptyset = \frac{\partial\emptyset}{\partial x}\mathbf{i} + \frac{\partial\emptyset}{\partial y}\mathbf{j} + \frac{\partial\emptyset}{\partial z}\mathbf{k}$$

Class Activity

If $\emptyset(x, y, z) = 2xz^4 - x^2y$, find $\mathbf{grad}\ \emptyset$ and $\|\nabla\emptyset\|$at $P(2,2,1)$

Directional Derivatives

Definition:

The directional derivative of $\emptyset(x, y, z)$ at any point P along the curve, denoted by $d\emptyset/ds$, is defined as

$$\frac{d\emptyset}{ds} = \lim_{\Delta s \to 0} \frac{\Delta\emptyset}{\Delta s}$$

if the limit exists.

Theorem: If $D_u\emptyset$ denotes the directional derivatives of ∅ in the direction of the unit vector $\mathbf{u}$, then

$$D_u\emptyset = \nabla\emptyset \cdot \mathbf{u}$$

By the geometrical property of the dot product,

$$\begin{aligned} \nabla\emptyset &= \|\nabla\emptyset\| \, \|\mathbf{u}\| \cos\theta \\ &= \|\nabla\emptyset\| \cos\theta \end{aligned}$$

Thus $max\{D_u\emptyset\} = \|\nabla\emptyset\|$
$min\{D_u\emptyset\} = -\|\nabla\emptyset\|$

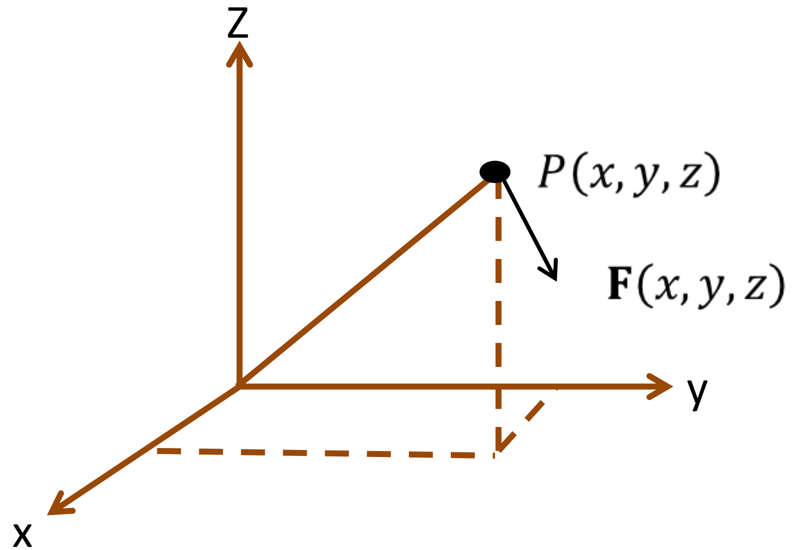
Class Activity

The electrical potential V at the point $P(x, y, z)$ is given by $V = x^2 + 4y^2 + 9z^2$. Find the rate of change of V at $P(6, -3, 2)$ in the direction from P to the origin. Find the direction that produces the maximum rate of change of V at P. What is the maximum rate of change at P?

Vector Fields

Definition:

If to each point $P(x, y, z)$ of a region in space there is made to correspond to a vector quantity $\mathbf{F}(x, y, z)$, then $\mathbf{F}(x, y, z)$ is a vector function and we say there is a vector field in the region.



Divergence of Vector Fields

Divergence of vector fields is a scalar field to describe the extent to which the field diverges from a point.

Definition: Let

$$\mathbf{F}(x,y,z) = \mathrm{F}_1(x,y,z)\mathbf{i} + \mathrm{F}_2(x,y,z)\mathbf{j} + \mathrm{F}_3(x,y,z)\mathbf{k}$$

be a vector function that is defined and differentiable in a region. Then the divergence of $\mathbf{F}$ is a scalar field, denoted by $\mathrm{div}\,\mathbf{F}$, defined by

$$\mathrm{div}\,\mathbf{F} = \mathbf{\nabla}\cdot\mathbf{F} = \frac{\partial \mathrm{F}_1}{\partial x} + \frac{\partial \mathrm{F}_2}{\partial y} + \frac{\partial \mathrm{F}_3}{\partial z}$$

Class Activity

Gives $\mathbf{F}(x, y, z) = xe^{y}\,\mathbf{i} + e^{xy}\,\mathbf{j} + \sin yz\;\mathbf{k}$

Find $\operatorname{div}\mathbf{F}$ at $(1,0,3)$.

Curl of Vector Fields

Definition: Let

$\mathbf{F}(x,y,z) = \mathrm{F}_1(x,y,z)\mathbf{i} + \mathrm{F}_2(x,y,z)\mathbf{j} + \mathrm{F}_3(x,y,z)\mathbf{k}$ be a vector function that is defined and differentiable in a region. Then the curl of the vector $\mathbf{F}$ is a vector field, denoted by $\mathbf{curl\ F}$, defined by

$$\mathbf{curl\ F} = \nabla \times \mathbf{F}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1(x,y,z) & F_2(x,y,z) & F_3(x,y,z) \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$$

Class Activity

Find **curl F** at point $(2, -3, 4)$ for $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + xz\,\mathbf{k}$

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