SSCE1993 ENGINEERING MATHEMATICS

MULTIPLE INTEGRALS

DR RASHIDAH BINTI AHMAD
ASSOC PROF DR MUNIRA BINTI ISMAIL

DOUBLE INTEGRALS

DOUBLE INTEGRALS

Definition: Let $z = f(x, y)$ be any continuous function on a region R in the xy−plane. Double integral of f over R is defined by

$$\iint_R f(x, y)\, dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta A_i$$

if the limit exists.

Iterated Integrals over Rectangular Region

Definition: If $R = \{(x,y): a \le x \le b, c \le y \le d\}$ and $f(x,y)$ is continuous in the rectangular region, then

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)dx\right] dy$$

$$\int_a^b \int_c^d f(x,y)dy\,dx = \int_a^b \left[\int_c^d f(x,y)dy\right] dx$$

Example: Evaluate the iterated integral.

$$\int_0^4 \int_0^3 y\,dx\,dy$$

Solution: $\int_0^4 \int_0^3 y\,dx\,dy = \int_0^4 \left[\int_0^3 y\,dx\right] dy$

$$= \int_0^4 [xy]_{x=0}^{x=3} dy$$

$$= \int_0^4 3y\,dy$$

$$= \left.\frac{3y^2}{2}\right|_0^4$$

$$= 24$$

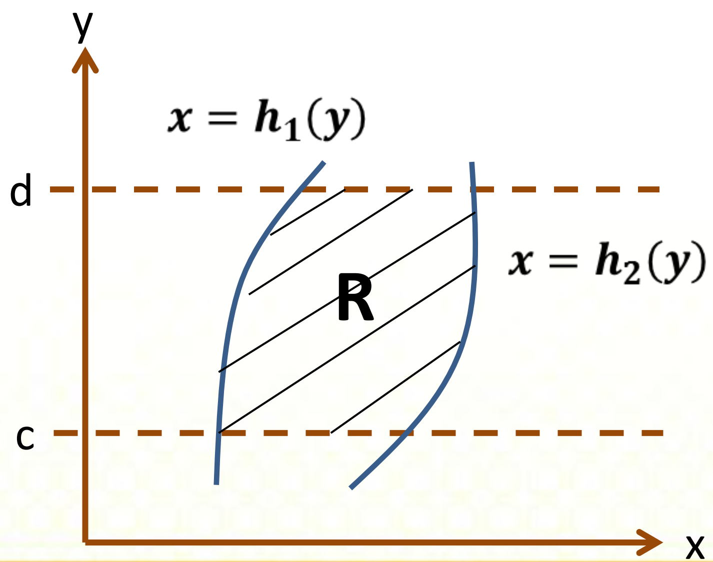
Iterated Integrals over Non Rectangular Region

Integrating first with respect to x

Let $f(x, y)$ be continuous on R where $R = \{(x, y): h_1(y) \leq x \leq h_2(y)\}, c \leq y \leq d$ with $h_1(y)$ and $h_2(y)$ continuous on $[c, d]$ then

$$\iint_R f(x, y)\, dA$$

$$= \int_c^d \int_{h_2(y)}^{h_1(y)} f(x, y) dx\, dy$$



Iterated Integrals over Non Rectangular Region

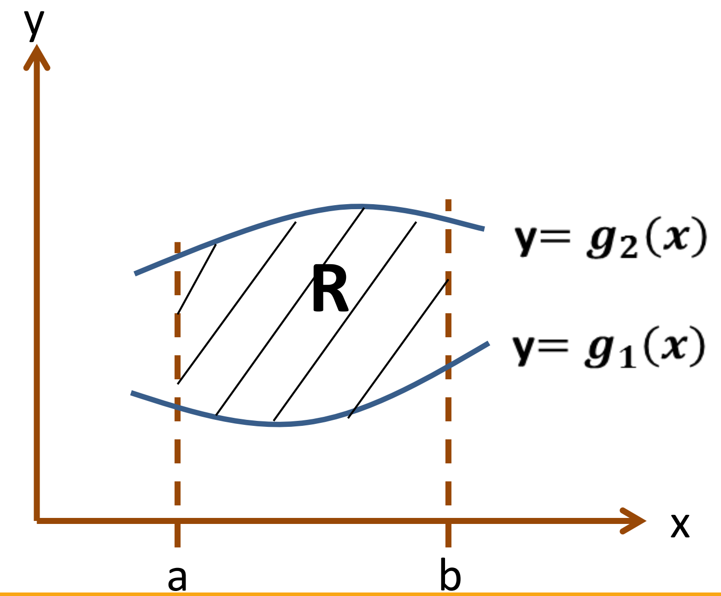
Integrating first with respect to y

Let $f(x,y)$ be continuous on R where

$$R = \{(x,y): a \le x \le b\,, g_1(x) \le y \le g_2(x)$$

with $g_1(x)$ and $g_2(x)$ continuous on $[a,b]$ then

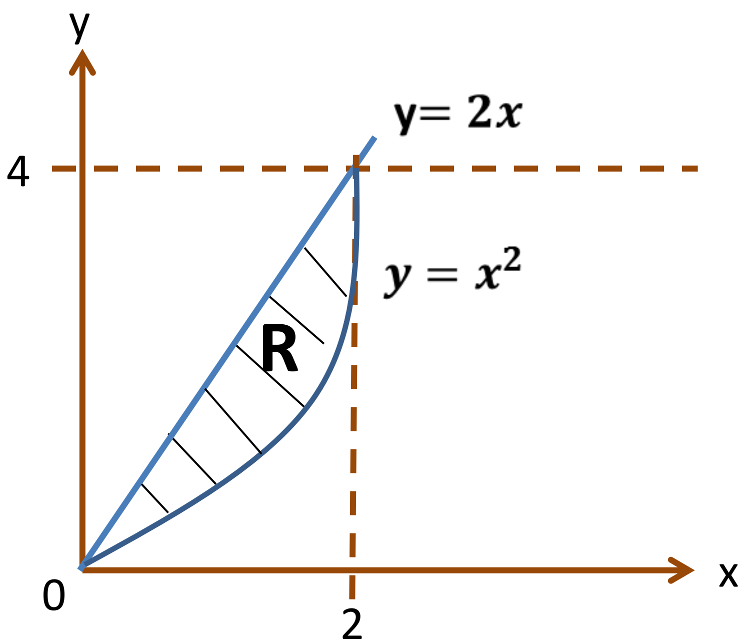
$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$



Example: Evaluate $\iint_R xy\,dA$ where R is the region bounded by $y = 2x$ and $y = x^2$ in the first quadrant.

Solution:

The region of integration.



Integrating with respect to y first.

$$\int_0^2 \int_{x^2}^{2x} xy\,dy\,dx = \int_0^2 \left[\frac{xy^2}{2}\right]_{x^2}^{2x} dx$$
$$= \int_0^2 \left(2x^3 - \frac{x^5}{2}\right) dx$$
$$= \frac{8}{3}$$

Integrating with respect to x.

$$\int_0^4 \int_{y/2}^{\sqrt{y}} xy\,dx\,dy = \int_0^4 \left[\frac{x^2y}{2}\right]_{y/2}^{\sqrt{y}} dy$$
$$= \int_0^4 \left(\frac{y^2}{2} - \frac{y^5}{8}\right) dx$$
$$= \frac{8}{3}$$

Double Integrals in Polar Coordinates

Changing from a Cartesian integral $\iint_R f(x,y)\,dA$ into polar coordinates.

Substitute $x = r\cos\theta\,, y = r\sin\theta$ and $d\mathrm{A} = r\,drd\theta$

$$\iint_R f(x,y)\ dA$$

$$= \iint_R f(r\cos\theta\,, r\sin\theta\,)r\ drd\theta$$

Example:

Evaluate $\iint_R (x^2 + y^2 + 1)dA$ where R is the region inside the circle $x^2 + y^2 = 4$

Double Integrals as Area and Volume

Theorem: If $f(x, y)$ is a continuous and nonnegative on a region R, then

i. The area of the region R is given by $\iint_R \, dA$

ii. The volume of the solid beneath the surface and above the region R is given by $\iint_R f(x, y) dA$

Class Activity

i) Find the area of the region enclosed by the parabola $y = x^2$ and the line $y = x + 6$

ii) Use double integration to find the volume of solid bounded by the cylinder $x^2 + y^2 = 4$ and the plane $x + z = 4$

TRIPLE INTEGRALS

Triple Integrals in Cartesian Coordinates

Theorem: Let $z = f(x, y, z)$ be any function that is continuous on a solid G where $G = \{x, y, z: a \leq x \leq b, c \leq y \leq d, k \leq z \leq l\}$ then

$$\iiint_G f(x, y, z)\, dV = \int_a^b \int_c^d \int_k^l f(x, y, z)\, dz\, dy\, dx$$

The order of integration in the iterated integral can be interchanged.

Class Activity

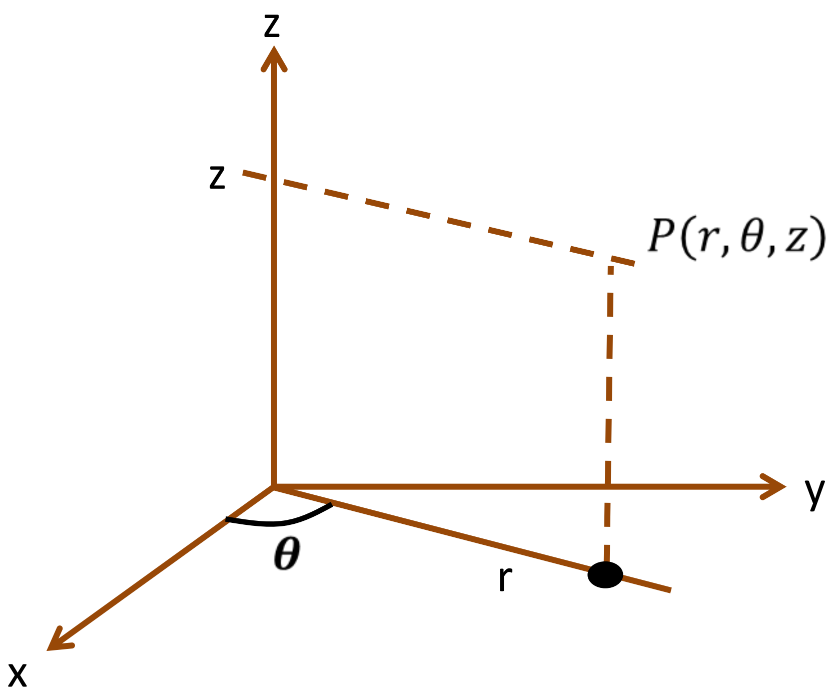
Suppose G is a solid in the first octant bounded by

$y = x^2,\ x + y = 1,\ xy$ –plane and yz –plane.

i. Evaluate $\iiint_G z\ dV$

ii. Find the volume of G

Triple Integrals in Cylindrical Coordinates



Relationship between Cartesian and cylindrical coordinates:

$$x = r\ cos\theta$$
$$y = r\ cos\theta$$
$$z = z$$
$$x^2 + y^2 = r^2, \qquad tan\theta = \frac{y}{x}$$

Triple Integrals in Cylindrical Coordinates

Let f be continuous function of r, θ and z on a bounded solid G. The triple integral of f over G is

$$\iiint_G f(x, y, z)\, dV$$

$$= \iint_R \int_{h(r,\theta)}^{g(r,\theta)} f(r, \theta, z)\ r\, dz\, dr\, d\theta$$

where R is the region in the xy $-$plane described by polar coordinates.

Class Activity

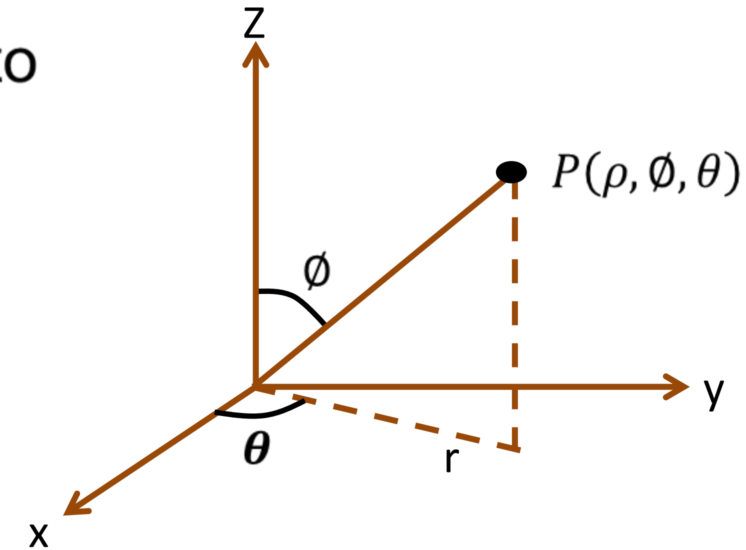
1. Use cylindrical coordinates to find the volume of the solid bounded by $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$

2. Use cylindrical coordinates to find the mass of the solid bounded by the surfaces $z = 4 - x^2 - y^2$, $z = x^2 + y^2$, $x^2 + y^2 = 1$ with density $\delta(x, y, z) = z$

Triple Integrals in Spherical Coordinates

Conversion formulas: Rectangular to spherical forms

$$x = \rho\, sin\emptyset cos\theta$$
$$y = \rho\, sin\emptyset sin\theta$$
$$z = \rho\, cos\emptyset$$
$$x^2 + y^2 + z^2 = \rho^2,$$



Let f be a continuous function of ρ, θ and ∅ on a bounded solid G , then the triple integrals of f over G

$$\iiint_G f(\rho,\emptyset,\theta)\, dV = \iiint_G f(\rho,\emptyset,\theta)\rho^2 sin\emptyset\;\, d\rho\, d\emptyset\, d\theta$$

Class Activity

1. Evaluate $\iiint_G \frac{dV}{\sqrt{x^2+y^2+z^2}}$ where G is the sphere $x^2 + y^2 + z^2 \leq 9$

2. Find the center of mass of the solid bounded by surface $z = \sqrt{x^2 + y^2}$ and the plane $z = 16$

REFERENCES

- Glyn James (2010). Advance Modern Engineering Mathematics, 4th Edition. Prentice Education Ltd.
- Howard Anton (2005). Multivariable Calculus, 8th Edition. John Wiley & Sons Inc.
- Kreysziq (2011). Advance Engineering Mathematics, 10th Edition. John Wiley & Sons Inc.
- Maslan Osman & Yusof Yaacob, 2008. Multivariable and Vector Calculus, UTM Press.
- Yusdariah, Roselainy & Sabariah. Multivariable Calculus for Indpt. Learners, Revised 2nd Ed. 2011. Pearson Educ. Pub