SSCE1993 ENGINEERING MATHEMATICS

SURFACE INTEGRALS

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SURFACE INTEGRAL

Surface Integral of a scalar field $\rho(x, y, z)$ **with respect to** surface area S, a part of σ, a surface in 3D over a region R in the xy -plane, and $\mathbf{N}$ is any outward unit normal to σ, is given by

$$\iint_{\sigma} \rho(x, y, z)\,dS$$

Evaluating Surface Integral with respect to surface area S , $\iint_{\sigma} \rho(x, y, z)dS$

is by evaluating the double integrals over the region R where $dS = |\mathbf{N}|dA$. If σ is given by $z = f(x, y)$, substitute this into the scalar field ρ , and the formula $|\mathbf{N}| = \sqrt{(f_x)^2 + (f_y)^2 + 1}$ then

$$\iint_{\sigma} \rho(x, y, z)dS = \iint_{R} \rho(x, y, f(x, y))|\mathbf{N}|dA$$

$$= \iint_{R} \rho(x, y, f(x, y))\sqrt{(f_x)^2 + (f_y)^2 + 1}dA$$

Class Activity: Write down the evaluation of Surface Integral with respect to surface area S when σ is given by : (a) $y = h(x,z)$ (b) $x = g(y,z)$

Class Activity: (a) Evaluate $\iint_{\sigma} x^2 y \, dS$ where σ is $y^2 + x^2 = a^2$ in the first octant between the planes $x = 0, x = 9, z = y$ and $z = 2y$.

(b) Evaluate $\iint_{\sigma} xyz \, dS$ where σ is $y^2 = x$ in the first octant between the planes $z = 0, z = 9, z = y$ and $y = 2$.

Application of Surface Integral: **FLUX**

is the measure of the **flow** of a vector field

$$\mathbf{F}(x,y,z)=M(x,y,z)\mathbf{i}+N(x,y,z)\mathbf{j}+R(x,y,z)\mathbf{k}$$

spreading onto a surface σ in the direction of an **outward unit normal** $\mathbf{n}$ to σ, is denoted and given by

$$\text{Flux}\,\mathbf{F}=\iint_{\sigma}\mathbf{F.n}\,dS$$

How to Evaluate Surface Integral $\iint_{\sigma} \mathbf{F}.\mathbf{n}\,dS$,

where $dS = |\mathbf{N}|\,dA$ and $\mathbf{n} = \dfrac{\mathbf{N}}{|\mathbf{N}|}$, is reduced to solving a double integrals of the scalar field $\mathbf{F}\cdot\mathbf{N}$ over the region R in the xy -plane defined by the surface σ :

$$\iint_{\sigma} \mathbf{F}.\mathbf{n}\,dS = \iint_{R} \mathbf{F}.\frac{\mathbf{N}}{|\mathbf{N}|}|\mathbf{N}|\,dA = \iint_{R} \mathbf{F}.\mathbf{N}\,dA$$

Class Activity: Find an outward normal vector **N** of a surface σ given by:

(a) $z = 4 - y^2 - x^2$

(b) $z = y^2 + x^2$

(c) $y = 4 - z^2$

(d) $y = z^2 + x^2$

(e) $x = 4 - y^2 - z^2$

(f) $x = z^2 + y^2$

Class Activity: Evaluate $\iint_{\sigma} \mathbf{F}.\mathbf{n}\, dS$ where $\mathbf{n}$ is the outward unit normal, σ is the portion the surface given by:

(a) $z+x+y=1$ in the first octant of the three dimensional coordinate system, while

$$\mathbf{F}(x,y,z)=(x+y)\mathbf{i}+N(y+z)\mathbf{j}+R(x+z)\mathbf{k} \; .$$

(b) $z=1-x^2-y^2$ that lies above the -plane in the three dimensional coordinate system, while

$$\mathbf{F}(x,y,z)=x\mathbf{i}+y\mathbf{j}+z\mathbf{k} \; .$$

Class Activity: Evaluate $\iint_{\sigma} \mathbf{F.n}\, dS$ where σ is a piecewise surface consisting of $\sigma_1, \sigma_2, \sigma_3$ joined together at some edges, then

$$\iint_{\sigma} \mathbf{F.n}\, dS = \iint_{\sigma_1} \mathbf{F.n}\, dS + \iint_{\sigma 2} \mathbf{F.n}\, dS = \iint_{\sigma 3} \mathbf{F.n}\, dS$$

Class Activity: Evaluate $\iint_{\sigma} \mathbf{F.n}\, dS$ where σ is a piecewise surface consisting of the planes $x = 3, x = 0, z = 0$ such that

$$0 \le x \le= 3, 0 \le y \le 2, 0 \le z \le 1$$

and $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$

Class Activity: Evaluate $\iint_{\sigma} \mathbf{F}.\mathbf{n}\, dS$ where σ is a closed surface enclosing a region of space or a solid G. Some example of a closed surface is the sphere or a closed cuboid.

Class Activity: Evaluate $\iint_{\sigma} \mathbf{F}.\mathbf{n}\, dS$ where is a closed cuboid such that

$$0 \le x \le= 3, 0 \le y \le 2, 0 \le z \le 1$$

and $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$

Divergence Theorem or Gauss theorem

If σ is a closed surface enclosing a solid G

$$\iint_{\sigma} \mathbf{F}.\mathbf{n}\, dS = \iiint_{G} \nabla \cdot \mathbf{F}\, dV$$

Where $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$

and $$\nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial R}{\partial z}$$

is the divergence of $\mathbf{F}$.

Class Activity: Use the divergence theorem to evaluate $\iint_\sigma \mathbf{F}.\mathbf{n}\, dS$ where σ is a closed surface:

(a) or cuboid defined as $0 \le x \le= 3, 0 \le y \le 2, 0 \le z \le 1$

and $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}.$

(b) Is the closed circular cylinder $x^2 + y^2 = 9$ between the planes $z = 2$ and $z = 0$

and $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^2\mathbf{k}.$

(c)Is the closed hemisphere $\sqrt{9 - x^2 - y^2}$ and $z = 0$ and $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}.$

Stoke's Theorem

If σ is an open surface whose boundary is a closed curve C in the three dimensional coordinate system, then

$$\oint_C \mathbf{F}.d\mathbf{r} = \iint_\sigma (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

Where the orientation of the normal vector $\mathbf{n}$ of σ and the direction of how $\mathbf{F}$ circulates around the boundary C whether clockwise or otherwise, obeys the right-hand rule.

Class Activity: following problems

(a) A hemisphere σ is given by $x^2 + y^2 + z^2 = 4$ that lies above the xy-plane and $\mathbf{n}$ is its outward unit normal vector with the vector field $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^2\mathbf{k}$.
Use Stokes theorem to evaluate $\iint_\sigma (\nabla \times \mathbf{F}).\mathbf{n}\, dS$

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(b) Use Stoke's theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = x^2\mathbf{i} + 4xy^3\mathbf{j} + y^2x\mathbf{k}$ and C is the rectangle on the plane $z = y$ whose vertices are

$$(0,-5,-5), (3,-5,-5), (3,5,5), (0,5,5)$$

clock direction if viewed from above the xy-plane.

Reference

- Glyn James (2010). Advanced Modern Engineering Mathematics, 4th Edition. Prentice Hall Pearson Education Ltd.
- Howard Anton(2005). Multivariable Calculus, 8th Edition. . John Wiley & Sons Inc.
- Kreyszig (2011). Advanced Engineering Mathematics, 10th Edition. John Wiley & Sons Inc.
- Maslan Osman & Yusof Yaacob, 2008. Multivariable and Vector Calculus, UTM Press.
- Yudariah, Roselainy & Sabariah. Multivariable Calculus for Indpt. Learners, Revised 2nd Ed. 2011. Pearson Educ. Pub.