SSCE1993 ENGINEERING MATHEMATICS

LINE INTEGRAL

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LINE INTEGRAL

Line Integral of a scalar field $f(x, y, z)$ with respect to arclength s, or distance travelled from a point $A(x_0, y_0, z_0)$ to the point $B(x_0, y_0, z_0)$, on a curve C in 3D given by the vector equation

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

is

$$\int_C f(x, y, z)\,ds$$

Evaluate Line Integral with respect to arclength

s , $\int_C f(x,y,z)ds$ from $A(x_0,y_0,z_0)$ to $B(x_0,y_0,z_0)$

If C is given by the vector equation

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

implies the parametric equations

$$x = x(t), y = y(t), z = z(t)$$

where $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$

and $|\mathbf{r}'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$

$$\int_C f(x,y,z)ds = \int_A^B f(x(t),y(t),z(t))|\mathbf{r}'(t)|dt$$

Class Activity: Evaluate the following line integrals with respect to arclength s

(a) $\int_C 1 + xy^2 \, ds$ where C is the curve in 2D given by

$$\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j}, \quad 0 \le t \le 1.$$

(b) $\int_C xy + z^3 \, ds$ where C is th curve in 2D given by $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ from $A(1,0,0)$ to $B(-1,0,\pi)$

Work Done As a Line Integral in xy -plane (2D)

Suppose a vector field $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ is acting on an object moving along a curve C in 2D from the point $A(x_0, y_0)$ to the point $A(x_n, y_n)$ and define the displacement vector $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$. Then **work done** or performed by the vector field $\mathbf{F}$ on that object is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{(x_0, y_0)}^{(x_n, y_n)} M(x,y)dx + N(x,y)dy$$

Class Activity: Evaluate the following line integrals in 2D

(a) $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(t) = y\mathbf{i} + xy^2\mathbf{j}$ and the path C is the curve (i) $y = x^2$ (ii) $x = y^2$ in 2D from the point (0,0) to (1,1).

(b) $\int_C x^2dx - xydy$ where the path C is a piecewise curve consisting of the shortest part from the point (0,2) to (2,0) on the circle of radius 2, center at the origin, and then on the line segment from (2,0) to (0,-2).

Properties of Line Integral

(a) Line Integral is dependent on its path

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F}\cdot d\mathbf{r} \neq \int_{C_3} \mathbf{F}\cdot d\mathbf{r}$$

(b) The order of line integral is from starting point A to destination point B and not in the order of the parametric increment of the given path.

$$\int_A^B \mathbf{F}\cdot d\mathbf{r} \neq \int_B^A \mathbf{F}\cdot d\mathbf{r}$$ In fact, $$\int_A^B \mathbf{F}\cdot d\mathbf{r} = -\int_B^A \mathbf{F}\cdot d\mathbf{r}$$

Properties of Line Integral

(c) For a piecewise curve consisting of $C = C_1 + C_2 + C_3$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r}$$

(d) Line integral on a closed path C is denoted by

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

Class Activity :Illustrate some properties of line integrals through the previous and the following examples.

(a) Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$

where $\mathbf{F}(x, y) = (x + y)\mathbf{i} + xy\mathbf{j}$ and C is a closed path consisting of line segments forming a triangle whose vertices are

$(2,0), (0,2)$ to $(-2,0)$ and back to $(2,0)$

Green's Theorem for Line Integral in 2D

Say a vector field in the xy -plane (2D) is given by

$$\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$$

Where the curl of $\mathbf{F}$ is defined by

$$\text{Curl}\,\mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \;;$$

and C is a **closed curve** which defines **a simply connected region** *R* enclosed in C, in the xy -plane. If Curl $\mathbf{F} \neq \mathbf{0}$, then line integral on can be found using Green Theorem, the formula:

$$\oint_C M(x,y)\,dx + N(x,y)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA.$$

Class Activity :Use Green Theorem to evaluate the following line integral in 2D.

(a) $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = (x+y)\mathbf{i} + xy\mathbf{j}$ and C is a closed path consisting of line segments forming a triangle whose vertices are $(2,0), (0,2)$ to $(-2,0)$ and back to $(2,0)$.

Class Activity :Use Green Theorem to evaluate the following line integral in 2D.

(b) $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = (e^x - y^3)\mathbf{i} + (\cos y + x^3)\mathbf{j}$ and C is the unit circle in the anticlockwise direction.

(c) $\oint_C x^2 y\,dx + x\,dy$ where C is the triangular path starting at the origin to $(1,0)$ to $(1,2)$.

Line Integral in xyz -space (3D)

Suppose a vector field $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j} + R(x,y,z)\mathbf{k}$ is acting on an object moving along a curve C in 3D from the point $A(x_0, y_0, z_0)$ to the point $A(x_n, y_n, z_n)$ and define the displacement vector $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$. Then the **work done** or performed by the vector field $\mathbf{F}$ on that object is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{(x_0,y_0,z_0)}^{(x_n,y_n,z_n)} M(x,y,z)dx + N(x,y,z)dy + R(x,y,z)dz$$

Class Activity: Evaluate the following line integrals in 3D

(a) $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(t) = yz\mathbf{i} - xz\mathbf{j} + xy\mathbf{k}$ and the path C is $\mathbf{r}(t) = e^{t}\mathbf{i} + e^{3t}\mathbf{j} + e^{-t}\mathbf{k}$ for $0 \le t \le 1$.

(b) $\int_C x^3 dx + xy dy + z^3 dz$ where the path C is $x = \sin t, y = \cos t, z = t^2$ for $0 \le t \le \pi/2$.

(c) $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(t) = (y+z)\mathbf{i} + (x-z)\mathbf{j} + (x+y)\mathbf{k}$ and the path C is the line segment from $A(1,2,3)$ to $B(4,0,5)$ to $C(-4,3,0)$.

CONSERVATIVE VECTOR FIELD

A vector field $\mathbf{F}$ is said to be **conservative** *if and only if*

(a) $\Delta \times \mathbf{F} = \mathbf{0}$

(b) $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C

(c) $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_3} \mathbf{F} \cdot d\mathbf{r}$

(d) There exist a **potential function** $\phi(x, y, z) = k$ so that $\nabla \phi = \mathbf{F}$ and k is any arbitrary constant.

The fundamental theorem for line integral f

If the vector field $\mathbf{F}$ is conservative and there exist the potential function is $\phi(x, y, z) = k$

Then the value of the line integral from an initial point $A(x_0, y_0, z_0)$ to destination point $A(x_n, y_n, z_n)$ is

$$\int_A^B \mathbf{F}\cdot d\mathbf{r} = \phi_B - \phi_A = \phi(x_0, y_0, z_0) - \phi(x_n, y_n, z_n).$$

Class Activity: Solve the following problems by using the fundamental theorem of line integrals

(a) Say $\mathbf{F} = 2xy^3\mathbf{i} + (1+3x^2y^2)\mathbf{j}$ is conservative. Find its potential function and hence evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the line segment

(i) from (2,3) to (-1,2)

(ii) from (2,3) to (0,0) to (-1,2)

(iii) from (2,3) to (0,0) to (-1,2) back to (2,3)

Class Activity: Potential Problem

(b) Say the vector field

$$\mathbf{F} = (e^x \cos y + yz)\mathbf{i} + (xz - 1 + e^x \sin y)\mathbf{j} + (xy + z)\mathbf{k}$$

Is acting on an object. If $\nabla\phi = \mathbf{F}$, find the potential function and find work done to move this object from point $(0,0,0)$ to point (π, π, π)

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