SSCE1993 ENGINEERING MATHEMATICS

PARTIAL DERIVATIVES

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Say $y = f(x)$ is a single variable function.
What is the **derivative** of f with respect to x?

The derivative of f with respect to x is denoted and defined by

$$\frac{dy}{dx} = f'(x) = \lim_{h \to \infty} \frac{f(x+h) - f(x)}{h}.$$

Which tells us how the value of y changes if x is allowed to vary.

Class Activity: (a) Recall the standard differentiation formulae with chain rule for some common single variable functions. (b) Find the derivative of the following functions:

$(a)\quad x^2,$

$(b)\quad \sin 3x,$

$(c)\quad 3\sin 3x.$

$(d)\quad \sin^2 x^2,$

$(e)\quad x^2 \sin 3x,$

$(f)\quad \dfrac{x^2+1}{\sin 3x}.$

Say $z = f(x, y)$ is a two variable function. What is the **partial derivative** of f with respect to x ?

The partial derivative of f with respect to x is denoted and defined by

$$\frac{\partial z}{\partial x} = f_x = \lim_{h \to \infty} \frac{f(x+h, y) - f(x, y)}{h}$$

Which tells us how the value of z changes if y is held fixed and x is allowed to vary.

Since $z = f(x, y)$ is a two variable function, also, what about the **partial derivative** of f with respect to y ?

Similarly, the partial derivative of f with respect to is denoted and defined by

$$\frac{\partial z}{\partial y} = f_y = \lim_{h \to \infty} \frac{f(x, y+h) - f(x, y)}{h}$$

Which tells us how the value of z changes if x is held fixed and y is allowed to vary.

How to compute partial derivatives?

In practice, the usual method is to hold one variable fixed treating it as a constant, and then differentiating the resulting function with respect to the variable that is allowed to vary by using the same standard formulas or derivative rules for functions of one variable.

If $z = f(x, y)$ then the **first order partial derivatives** are $\dfrac{\partial z}{\partial x} = f_x$ and $\dfrac{\partial z}{\partial y} = f_y$

Class Activity: Compute the first order partial derivatives for the following functions of two variables.

$$(a)\quad f(x,y) = x^3y + xy^2,$$

$$(b)\quad f(r,s) = \ln r \sin(3r+s),$$

$$(c)\quad P(v,t) = \frac{t}{\tan(3v+t)}.$$

If $w = f(x, y, z)$ is a three variable function, then its **first order partial derivatives** are

$$\frac{\partial w}{\partial x} = f_x, \frac{\partial w}{\partial y} = f_y \quad \text{and} \quad \frac{\partial w}{\partial z} = f_z$$

Which tells us how the value of w changes when two variables are held fixed and one variable is allowed to vary.

Class Activity: Compute the first order partial derivatives for the following functions of three variables.

$$(a) \quad f(x,y,z) = x^2 y + xy^2 z + e^{xz},$$

$$(b) \quad f(r,s,t) = r\cos s + z^2 \sin t,$$

Interpret the partial derivatives of $z = f(x, y)$ as **rate of change**.

$f_x(x_0, y_0)$ is the rate of change in z when x changes from x_0 and y is fixed at y_0 .

$f_x(x_0, y_0)$ is the rate of change in z when y changes from y_0 and x is fixed at x_0 .

Class Activity: Compute the following rate of change by using partial derivatives.
Given $I = E/R$.

(a) Find the rate of change of current I with respect to the voltage E if resistance R is fixed at 15 ohms when E is 120 volts.

(b) Find the rate of change of current I with respect to the resistance R if voltage E is fixed at 120 volts and when R is 15 ohms.

Class Activity: For a real gas, Van der Waal's equation states that

$$\left(P+\frac{n^2a}{v^2}\right)(v-ab)=nRT$$

where P is pressure, v volume, T temperature and the rest of the symbols a,b,n are some constants. Compute and interpret $\frac{\partial P}{\partial v}$ and $\frac{\partial T}{\partial P}$ as a rate of change.

Interpret the partial derivative $f_x(x_0, y_0)$ of $z = f(x, y)$ as the **slope of a tangent line**.

$f_x(x_0, y_0)$ is the slope of the tangent line parallel to the xz –plane, at the point (x_0, y_0, z_0) that is also on the intersecting curve of the surface $z = f(x, y)$ with the fixed plane $y = y_0$.

Class Activity: Sketch a graph to show the interpretation of $f_x(x_0, y_0)$ as the slope of a tangent line.

Interpret the partial derivative $f_y(x_0, y_0)$ of $z = f(x, y)$ as the **slope of a tangent line**.

$f_y(x_0, y_0)$ is the slope of the tangent line parallel to the yz –plane, at the point (x_0, y_0, z_0) that is also on the intersecting curve of the surface $z = f(x, y)$ with the fixed plane $x = x_0$.

Class Activity: Sketch a graph to show the interpretation of $f_y(x_0, y_0)$ as the slope of a tangent line.

Class Activity: Interpret $f_x(1,3)$ for

$$f(x,y) = x\sqrt{x+y}$$

as a slope of a tangent line.

Higher Order Partial derivatives.

Say $z = f(x, y)$. The **second order partial derivatives** are obtained by computing partial derivatives of the first order partial derivatives f_x and f_y with respect to x and to y giving four different forms:

(a) $f_{xx} = \dfrac{\partial^2 f}{\partial x^2}$, (b) $f_{xy} = \dfrac{\partial^2 f}{\partial y \partial x}$, (c) $f_{yx} = \dfrac{\partial^2 f}{\partial x \partial y}$, (d) $f_{yy} = \dfrac{\partial^2 f}{\partial y^2}$

$$w = f(x, y, z)$$

Class Activity: (a) Disscuss the differences in these two types of notations. (b)Write the second order partial derivatives for $w = f(x, y, z)$.

(c) Answer the following questions on higher order partial derivatives:

(a) Given $f(x,y) = x^3 y + y^2 e^x$. Evaluate $\dfrac{\partial^2 f}{\partial x \partial y}(3,2)$.

(b) If $f(x,y) = x^2 \tan^{-1}\left(\dfrac{y}{x}\right)$. Find $f_{yx}(1,1)$.

(c) Say $f(x,y,z) = \cos(xy) + \sqrt{4 - zy}$. Find $\dfrac{\partial^4 f}{\partial x \partial y^2 \partial z}$.

Class Activity: (a) Recall the chain rule formula for single variable function $y = f(x)$ and with a single parameter $x = x(t)$:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

(b) Use the chain rule to find $\frac{dy}{dt}$ where $y = \sin(x)$ and $x = \cos(t)$.

(c) Introduce and sketch a tree diagram, to help in the construction of a chain rule formula.

Class Activity: Construct the chain rule for partial derivatives by using the tree diagram.

(a) Say $z = f(x,y)$ with one parameter $x = x(t), y = y(t)$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

(b) Say $z = f(x,y)$ With two parameters $x = x(\theta,t), y = y(\theta,t)$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}} \qquad \boxed{\frac{dz}{d\theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}}$$

(c) $w = f(x,y,z)$ With several parameters

$$x = x(r,s,t), y = y(r,s,t), z = z(r,s,t)$$

Class Activity: Solve the following using chain rule partial derivatives formula.

(a) Let $z=\sqrt{x^2+2xy},$ where $x=\cos(\theta), y=\sin(\theta)$

Find $\dfrac{dz}{d\theta}$.

(b) Given $f(x,y)=x^2-y^2$ where $x=u\sin v, y=u-2v$

Find $\dfrac{\partial f}{\partial u}, \dfrac{\partial f}{\partial v}$.

(c) Find $\dfrac{\partial w}{\partial z}$ if $w=r^2+sv+t^3$ where $r=x^2+y^2+z^2,$

$s=xyz, v=e^y, t=yz^2.$

Class Activity: Solve the following using chain rule partial derivatives formula.

(e) If $z = f(x, y)$ where $x = r\cos(t)$ and $y = r\sin(t)$

Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ and show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial t}\right)^2$$

(f) If $z = f(x - y)$ show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0.$

Implicit Partial Differentiation For Single Variable Function.

If $y = f(x)$ is defined implicitly by the equation $F(x, y) = 0$

It can be proven by the chain rule that $\dfrac{\partial y}{\partial x} = -\dfrac{F_x}{F_y}$

Where F_x, F_y are the first partial derivatives of F provided that $F_y \neq 0$.

Class Activity: Find $\dfrac{dy}{dx}$ where $y = f(x)$ given implicitly by

(a) $\sin(x+y) + \cos(x-y) = y$ (b) $x^2 y + \sqrt{xy} = 4$

(c) $xy^2 + \ln(2x+y) = 5$ (d) $x\cos y + y\tan^{-1} x = x$.

Implicit Partial Differentiation For Two Variable Funtions.

If $z = f(x,y)$ is defined implicitly by the equation $F(x,y,z) = 0$
It can be proven by the chain rule that

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Where F_x, F_y, F_z are the first partial derivatives of F
provided that $F_z \neq 0$.

Class Activity: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where $z = f(x,y)$ given implicitly by

(a) $xyz = \cos(x+y+z)$ (b) $y^5 + y^3 z^2 = 1 + e^{x^2 z}$

(c) $\ln(x+yz) = 4 + xy^2 z^3$.

Differential For Approximation.

Say $y = f(x)$ and x changes by Δx at a given point x which implies that y will change exactly by Δy. We can approximate Δy by the **differentials** df, that is

$$\boxed{\Delta y \approx df}$$

Where $df = f'(x)\delta x$ and $\delta x = \Delta x$ and $f'(x) = \frac{dy}{dx}$.

Class Activity: Use differentials to approximate the change in volume of a spherical balloon when its radius changes from 3 cm to 3.15 cm.

Total Differentials For Approximation.

Say $z = f(x, y)$ and x changes by Δx while y changes by Δy at a given point (x, y) which implies that z will change exactly by Δz . We can approximate Δz by the **total differentials** df that is

$$\boxed{\Delta z \approx df}$$

Whose formula is $df = f_x \delta x + f_y \delta y$ and $\delta x = \Delta x,\ \delta y = \Delta y$

and f_x, f_y are the first partial derivatives of f with respect to x and y respectively.

Class Activity: (a) Say a cylinder has height 8 cm and radius 3 cm. After an experiment, the height increases to 8.12cm but the radius decreases to 2.95 cm. Approximate the change in volume of the cylinder by using total differentials (or partial derivatives). What is the possible maximum change aprroximation in volume?

(b) The height and radius of a cylinder is measured with an error of 2% and 4% respectively. What is the maximum percentage error in the calculation of its volume?

(b) A gas flow through a pipe is given by the formula

$$V(r,t) = Cr^{\frac{1}{2}}t^{-\frac{5}{6}}$$

where C is constant, and the maximum error in measuring r and t is 1.6% and 0.36% respectively. What is the maximum percentage error in the calculating V ?

Local Extrema of $y = f(x)$.

Say $y = f(x)$. $x = a$ is a critical point if $f'(a) = 0$ or if it doesn't exist..

Say $x = a$ is a critical point of $y = f(x)$. $x = a$ is a **local maximum** if $f(a) \geq f(x)$
and is a **local minimum** if $f(a) \leq f(x)$
for every point x near $x = a$.
$x = a$ is a reflexive point if $f'(a)$ does not exist.

Class Activity: Sketch a graph of $y = f(x)$ that show example of some critical points whether local maximum or minimum, or reflexive point .

Local Extrema of $z = f(x, y)$.

Class Activity: Sketch the following graphs of $z = f(x, y)$ and point out their critical points:

(a) $z = x^2 + y^2$

(b) $z = 4 - x^2 - y^2$

(c) $z = y^2 - x^2$

The First Partial Derivative Test .

(a,b) is a critical point of $z = f(x,y)$ if

(a) both the first partial derivatives are zero:

$$f_x(a,b) = 0 \text{ and } f_y(a,b) = 0$$

(b)or if one of these first partial derivatives above does not exist at (a,b)

Say (a,b) is a critical point of $z = f(x,y)$. The point (a,b) is a **local extremum** if $f(a,b) \geq f(x,y)$ hence (a,b) is a **local maximum** and is a **local minimum** if $f(a,b) \leq f(x,y)$ for every point (x,y) near (a,b) .

(a,b) is a **saddle** point if the traces of the graph on vertical planes through this point has a local maximum on one and a local minimum on the other.

The Second Partial Derivative Test .

Say (a,b) is a critical point of $z = f(x,y)$ where all the second partial derivatives exist at (a,b) . Let $D(x,y)$ be the determinant of the second partial derivative test given by

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

If (a) $D(a,b) > 0 \Rightarrow (a,b)$ is a local extremum

(b) $D(a,b) < 0 \Rightarrow (a,b)$ is a saddle

(c) $D(a,b) = 0 \Rightarrow$ no conclusion can be made

Further if (a,b) is a local extremum, then if

(d) $f_{xx}(a,b) > 0 \Rightarrow (a,b)$ is a local minimum

(e) $f_{xx}(a,b) < 0 \Rightarrow (a,b)$ is a local maximum

Class Activity: Find all critical points for the following function and classify each critical point as a local maximum, local minimum or a saddle point.

(a) $f(x, y) = 4xy - x^4 - y^4$

(b) $f(x, y) = x^2 y - 3x^2 - 6y^2$

(c) $f(x, y) = x^3 - 3xy + y^3$

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