SEE 2523 Theory Electromagnetic

Chapter 3
Theorems, Potential Energy, Laplace’s & Poisson’s Equations

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Derivation of Laplace's and Poisson's Formulations

Maxwell's Equations

1. Modern electromagnetism is based on four fundamental relations

Gauss's Law

$$\vec{\nabla} \cdot \vec{D} = \rho_v$$

Gauss's Law

$$\vec{\nabla} \cdot \vec{B} = 0$$

Faraday's Law

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Ampere's Law

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

where $\vec{E}$ is the electric field,

$\vec{H}$ is the magnetic field,

$\vec{D}$ is the electric flux density or electric displacement,

$\vec{B}$ is the magnetic flux density,

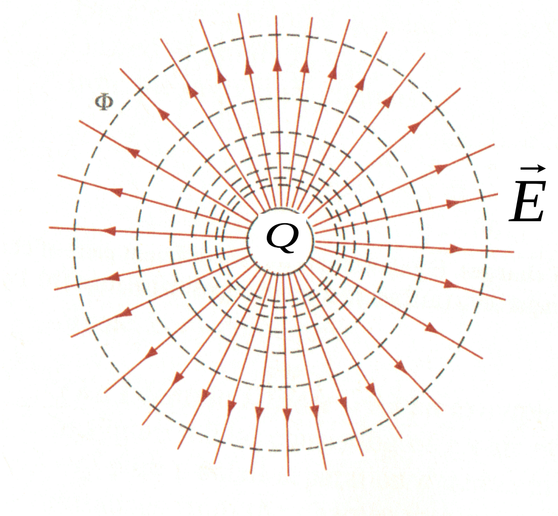
$\vec{J}$ is the current density,

ρv is the charge density.

From Gauss's law, the Coulomb's law has been derived.

Flux Density (1)

1. Electric flux line is an imaginary path or line drawn in a direction at any point (Direction of the electric field, $\vec{E}$ at that point).



2. The electric flux, ψ is defined as

$$\psi = \int_S \vec{D} \cdot d\vec{S}$$

where $\vec{D}$ is the electric flux density or so-called electric displacement.

Flux Density (2)

3. The electric flux density, $\vec{D}$ is defined as

$$\vec{D} = \varepsilon_o \vec{E}$$

4. The flux show the electric field intensity, $\vec{E}$ is dependent on the medium in which the charge is placed.

5. Gauss's law state that the electric flux, ψ passing through any closed surface is equal to the total charge, *Q* enclosed by that surface.

$$Q = \oint_S \vec{D} \cdot d\vec{S}$$

or

$$Q = \int_v \vec{\nabla} \cdot \vec{D} \; dv$$

Del Operator (1)

1. The del operator, $\vec{\nabla}$ is the vector differential operator.

2. In Cartesian coordinates

$$\vec{\nabla} = \frac{\partial(\;)}{\partial x}\hat{x} + \frac{\partial(\;)}{\partial y}\hat{y} + \frac{\partial(\;)}{\partial z}\hat{z}$$

Gradient Operator

$$\vec{\nabla} V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}$$

$$\vec{\nabla} = \frac{\partial(\;)}{\partial x}\hat{x} + \frac{\partial(\;)}{\partial y}\hat{y} + \frac{\partial(\;)}{\partial z}\hat{z}$$

Divergence Operator

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Del Operator (2)

3. In cylindrical coordinates

$$\vec{\nabla} = \frac{\partial(\ )}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial(\ )}{\partial\phi}\hat{\phi} + \frac{\partial(\ )}{\partial z}\hat{z}$$

Gradient Operator

$$\vec{\nabla}V = \frac{\partial V}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial V}{\partial\phi}\hat{\phi} + \frac{\partial V}{\partial z}\hat{z}$$

$$\vec{\nabla} = \frac{1}{\rho}\frac{\partial\rho(\ )}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial(\ )}{\partial\phi}\hat{\phi} + \frac{\partial(\ )}{\partial z}\hat{z}$$

Divergence Operator

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\frac{\partial\rho A_\rho}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

Del Operator (3)

4. In spherical coordinates

$$\vec{\nabla} = \frac{\partial(\ )}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial(\ )}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial(\ )}{\partial \phi}\hat{\phi}$$

Gradient Operator

$$\vec{\nabla} V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}$$

$$\vec{\nabla} = \frac{1}{r^2}\frac{\partial\left(r^2\ \ \right)}{\partial r}\hat{r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta\ \ \right)}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial(\ )}{\partial \phi}\hat{\phi}$$

Divergence Operator

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial\left(r^2 A_r\right)}{\partial r}\hat{r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta\, A_\theta\right)}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}\hat{\phi}$$

Del Operator (Gradient Operator) (2)

5. The operator has no physical meaning by it self.

6. The operator attains a physical meaning once it operates on a scalar physical quantity.

7. The result of the operation is a vector whose magnitude is equal to the maximum rate of change of the physical quantity per unit distance.

8. The direction result of the operation is along the direction of maximum increase.

Del Operator (Gradient Operator) (3)

9. The operator is useful in defining

a) Gradient of a scalar *V*

$\vec{\nabla} V$ → Electric field intensity, Potential

b) Divergence of a vector **A**

$\vec{\nabla} \cdot \vec{A}$ → Electric flux, Magnetic flux

c) Curl of a vector **A**

$\vec{\nabla} \times \vec{A}$ → Magnetic field intensity, Current

d) Laplacian of a scalar *V*

$\vec{\nabla}^2 V$ → Stored energy

UTM

Mathematical Theorem

1. There are two important mathematical theorem

   a) Gauss's Theorem (Divergence Theorem)

   b) Stoke's Theorem (Curl)

2. The divergence theorem states that the total outward flux of a vector field $\vec{A}$ through the closed surface ***S*** is the same as the volume integral of the divergence $\vec{A}$

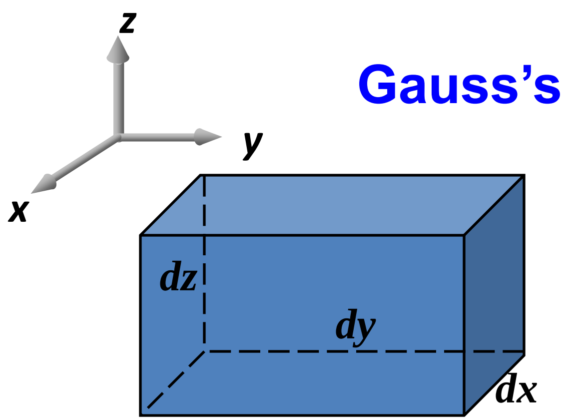
$$\oint_S \vec{A} \cdot d\vec{S} = \int_v \vec{\nabla} \cdot \vec{A} \, dv$$

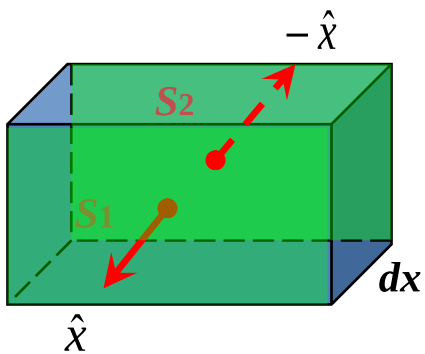
3. The Stoke's theorem states that the circulation of a vector field $\vec{A}$ around a closed path ***L*** is equal to the surface integral of the curl of $\vec{A}$ over the open surface ***S*** bounded by ***L***.

$$\oint_L \vec{A} \cdot d\vec{l} = \int_S \left(\vec{\nabla} \times \vec{A}\right) \cdot d\vec{S}$$

$\vec{A}$ and $\vec{\nabla} \times \vec{A}$ are continuous on ***S***

Gauss's Theorem (Divergence Theorem) (1)





Consider an element of volume $dv = dxdydz$ in a vector field, $\vec{F}$

For surface S_1

$$d\vec{S}_1 = \hat{x}\,dy\,dz$$

and the vector field, $\vec{F}$ though the surface S_1

$$\vec{F}\cdot d\vec{S}_1 = \left(F_x\hat{x} + F_y\hat{y} + F_z\hat{z}\right)\cdot\hat{x}\,dy\,dz$$
$$= F_x\,dydz$$

For surface S_2

$$d\vec{S}_2 = -\hat{x}\,dy\,dz$$

and the vector field, $\vec{F}$ though the surface S_2

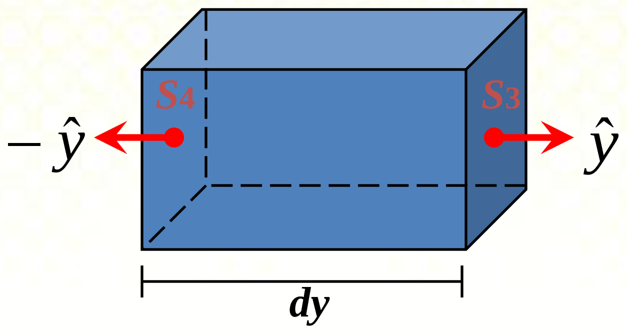
$$\vec{F}\cdot d\vec{S}_2 = \left(F'_x\hat{x} + F'_y\hat{y} + F'_z\hat{z}\right)\cdot -\hat{x}\,dy\,dz$$
$$= -F'_x\,dydz$$

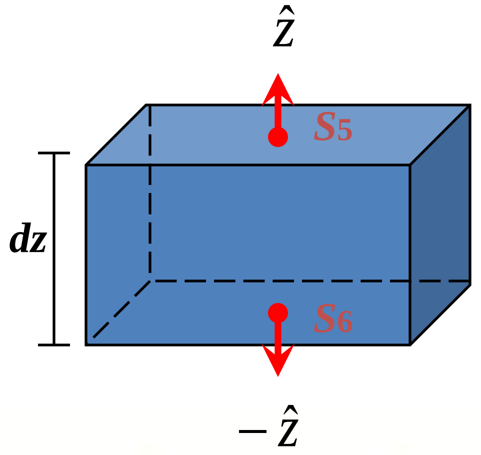
Combining these two results

$$\vec{F}\cdot d\vec{S}_1 + \vec{F}\cdot d\vec{S}_2 = F_x\,dy\,dz - F'_x\,dy\,dz$$
$$= \frac{\partial}{\partial x}\left(F_x\,dy\,dz\right)dx$$

$$\int_{S_1+S_2}\vec{F}\cdot d\vec{S} = \int_v\left(\frac{\partial F_x}{\partial x}\right)dx\,dy\,dz$$

Gauss's Theorem (Divergence Theorem) (2)





Similarly, for surface S_3 and S_4

$$\int_{S_3+S_4} \vec{F} \cdot d\vec{S} = \int_v \left(\frac{\partial F_y}{\partial y} \right) dx\, dy\, dz$$

and for surface S_5 and S_6

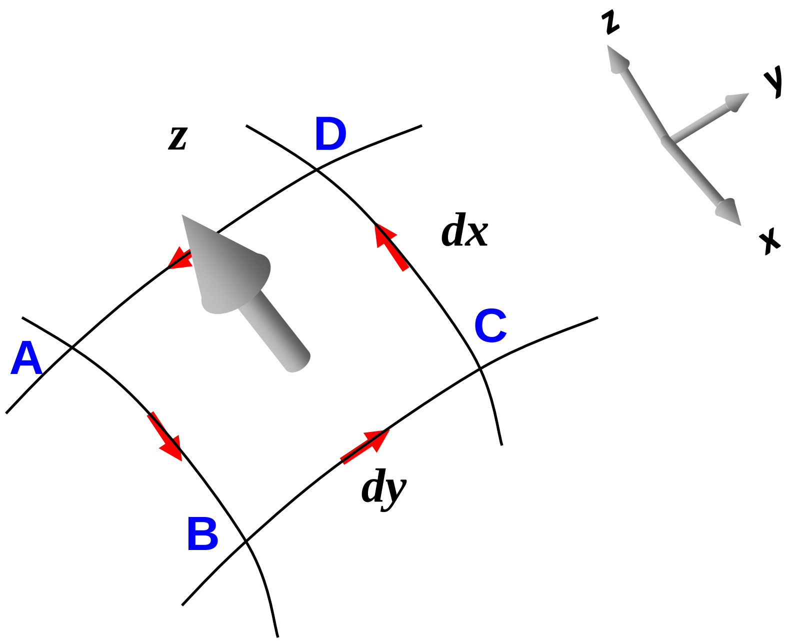
$$\int_{S_5+S_6} \vec{F} \cdot d\vec{S} = \int_v \left(\frac{\partial F_z}{\partial z} \right) dx\, dy\, dz$$

These three results together cover the total surface:

$$\int_{S_1+\ldots+S_6} \vec{F} \cdot d\vec{S} = \int_v \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dx\, dy dz$$
$$= \int_v \vec{\nabla} \cdot \vec{F}\, dv$$

Stoke's Theorem (1)

Let $\vec{F}_a$, $\vec{F}_b$, $\vec{F}_c$ and $\vec{F}_d$ denote the vector field at **A**, **B**, **C** and **D**, respectively.



The vector field along AB

$$\vec{F}_a \cdot \hat{x}\,dx = \left(F_{ax}\hat{x} + F_{ay}\hat{y} + F_{az}\hat{z}\right)\cdot\hat{x}\,dx$$
$$= F_{ax}\,dx$$

The vector field along BC

$$\vec{F}_b \cdot \hat{y}\,dx = \left(F_{bx}\hat{x} + F_{by}\hat{y} + F_{bz}\hat{z}\right)\cdot\hat{y}\,dy$$
$$= F_{by}\,dy$$

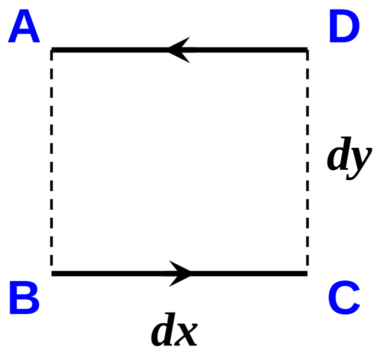
The vector field along DA

$$\vec{F}_d \cdot -\hat{y}\,dx = \left(F_{dx}\hat{x} + F_{dy}\hat{y} + F_{dz}\hat{z}\right)\cdot -\hat{y}\,dy$$
$$= -F_{dy}\,dy$$

The vector field along CD

$$\vec{F}_c \cdot -\hat{x}\,dx = \left(F_{cx}\hat{x} + F_{cy}\hat{y} + F_{cz}\hat{z}\right)\cdot - \hat{x}\,dx$$
$$= -F_{cx}\,dx$$

Stoke's Theorem (2)

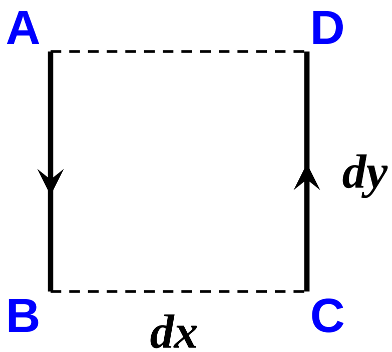


The total vector field for x-axis (**AB**+**CD**)

$$\int_{(AB+CD)} \vec{F}\cdot d\vec{l} = F_{ax}\,dx - F_{cx}\,dx$$

$$= -\left(F_{cx} - F_{ax}\right)dx$$

$$= -\frac{\partial F_x}{\partial y}\,dx\,dy$$



The total vector field for y-axis (**BC**+**DA**)

$$\int_{(BC+DA)} \vec{F}\cdot d\vec{l} = F_{by}\,dy - F_{dy}\,dy$$

$$= \left(F_{by} - F_{dy}\right)dy$$

$$= \frac{\partial F_y}{\partial x}\,dx\,dy$$

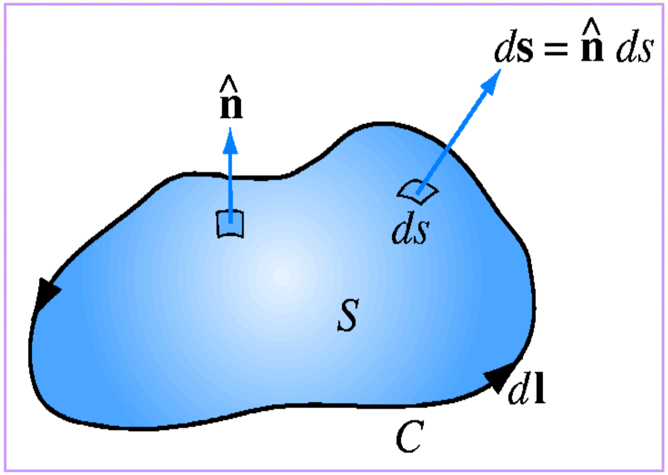
Adding these two results together for the complete rectangular (**ABCD**)

$$\int_{(ABCD)} \vec{F}\cdot d\vec{l} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)dx\,dy$$

Review back the cross product

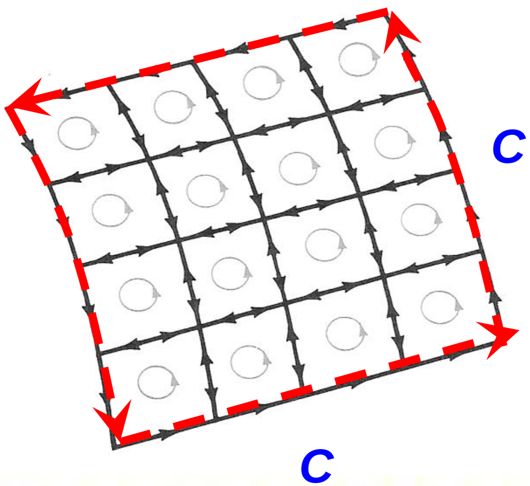
$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - \hat{y}\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right) + \hat{z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$



$$\int_{(ABCD)} \vec{F}\cdot d\vec{l} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dx\,dy$$

$$= \left(\vec{\nabla}\times\vec{F}\right)\cdot \hat{z}\, dx\,dy$$



Summing for all such elements over the surface

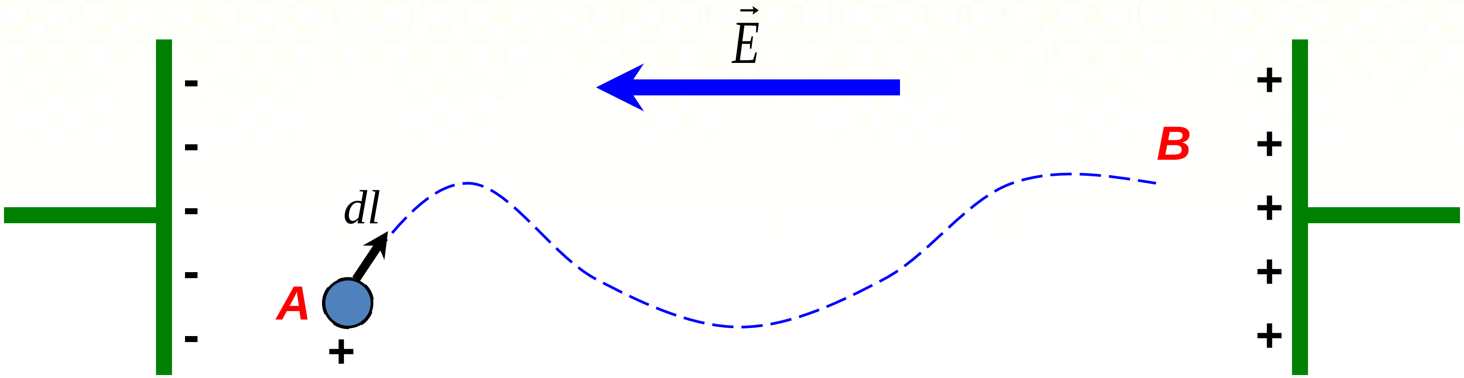
$$\int_S \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \sum\left(\int_{(ABCD)} \vec{F}\cdot d\vec{l}\right)$$

$$= \oint_C \vec{F}\cdot d\vec{l}$$

The vector field on the boundary lines between adjacent rectangular elements will cancel out, except on the boundary curve, ***C*** of the surface, ***S***

Electric Potential (Potential Energy)

1. **Potential energy is a energy required by charge particle to move in a region against an electric field, $\vec{E}$.**



2. **This is because work has to be done to overcome the force, $\vec{F}$ due to the electric field, $\vec{E}$ (negative sign).**

3. **The work done in displacing the charge by $d\vec{l}$ is**

$$dW = -\vec{F} \cdot d\vec{l}$$

4. **The work done in displacing from *A* to *B***

$$W = -\int_A^B \vec{F} \cdot d\vec{l}$$

Electric Potential (Work)

5. From Coulomb's law, the force on charge, Q is $\vec{F} = Q\vec{E}$, so the work done, W

$$dW = -\vec{F}\cdot d\vec{l} \\ = -Q\,\vec{E}\cdot d\vec{l}$$

6. Thus, the total work done, ***W*** from ***A*** to ***B***

$$W = -Q\int_A^B \vec{E}\cdot d\vec{l}$$

7. Potential at any point is the potential difference between that point and a chosen point (reference point) at which the potential is zero.

8. The potential between two points represents potential energy (work done) required to move a unit charge between the two points (points ***A*** and ***B***).

$$V_{AB} = \frac{W}{Q} \\ = -\int_A^B \vec{E}\cdot d\vec{l}$$

Electric Potential (Potential Difference)

9. V_{AB} is called **potential difference** between points ***A*** and ***B***.

10. **Potential difference**, V_{AB} is independent of the path between ***A*** and ***B***.

11. **Potential difference**, V_{AB} is measured in joules per coulomb, commonly referred to as volts **(V)**.

12. **Potential difference** between points ***A*** and ***B*** can be written as

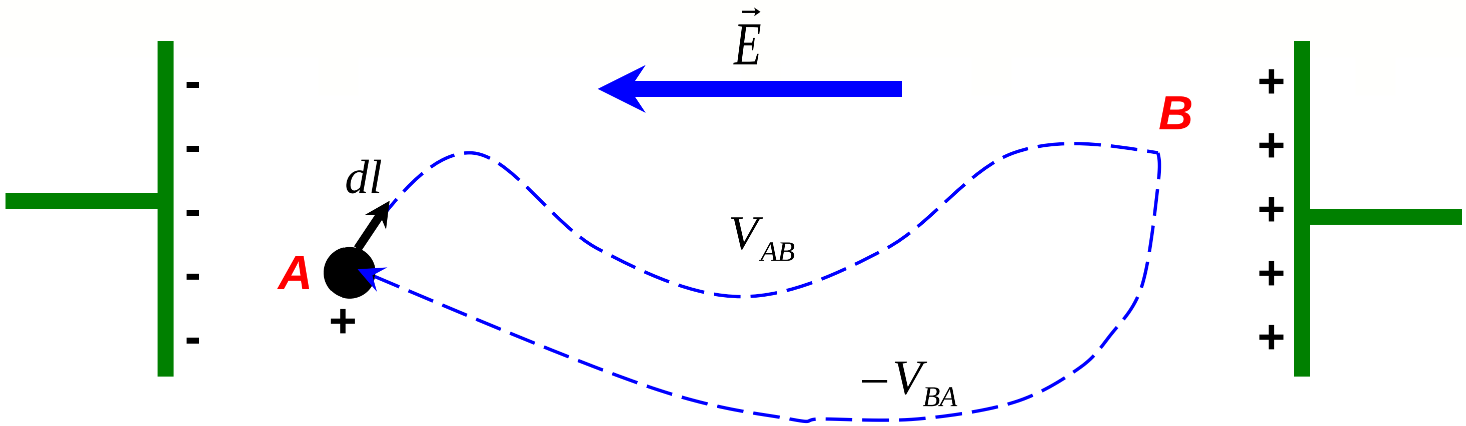
$$V_{AB} = V_B - V_A$$

where V_B and V_A are the **absolute potentials** at ***B*** and ***A***, respectively.

Electric Potential (Conservation of Energy)

1. **Potential difference is independent of the path between *A* and *B*.**

$$V_{AB} = -V_{BA}$$



Electric Potential (Conservation of Energy)

2. Electrostatic field is a conservative field.

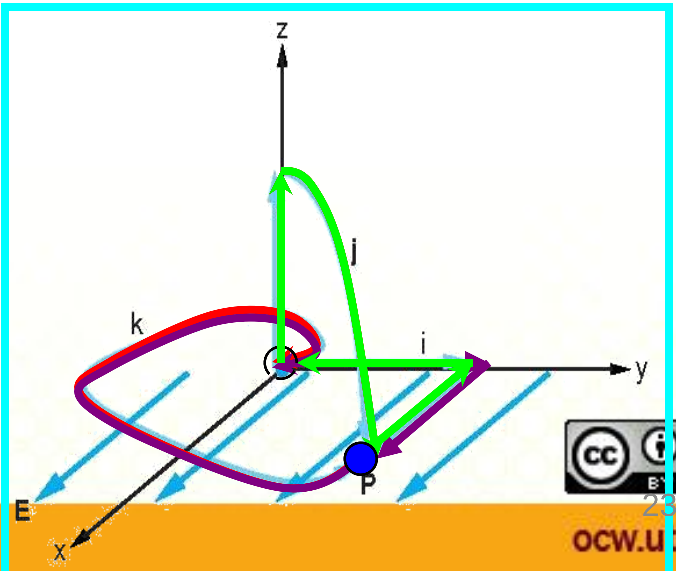
3. There are no net work is done in moving a charge along a closed path in an electrostatic field.

$$\oint_L \vec{E} \cdot d\vec{l} = 0$$

4. From Stokes's theorem, electrostatic field is conservative, or irrotational.

$$\vec{\nabla} \times \vec{E} = 0$$

5. Vector whose line integral does not depend on the path of integration are called conservative vectors.

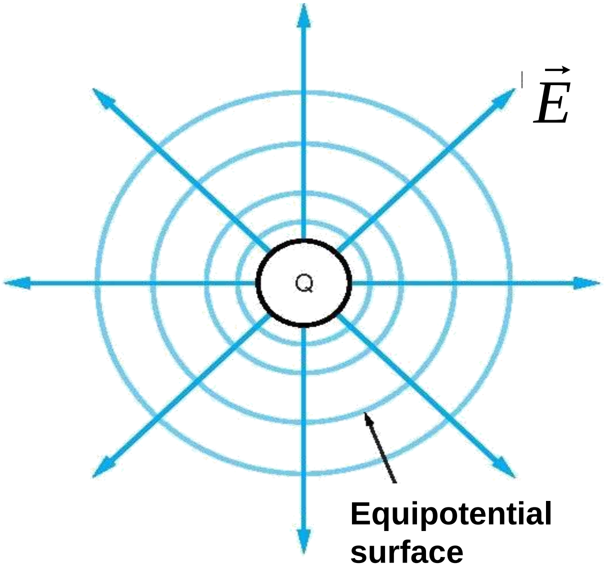


Electric Potential (Gradient)

1. The potential contours from a point charge form equipotential surfaces surrounding the point charge.

2. The surfaces are always orthogonal to the field lines.

3. The electric field, $\vec{E}$ can be determined by finding the maximum rate and direction of spatial change of the potential field, V .



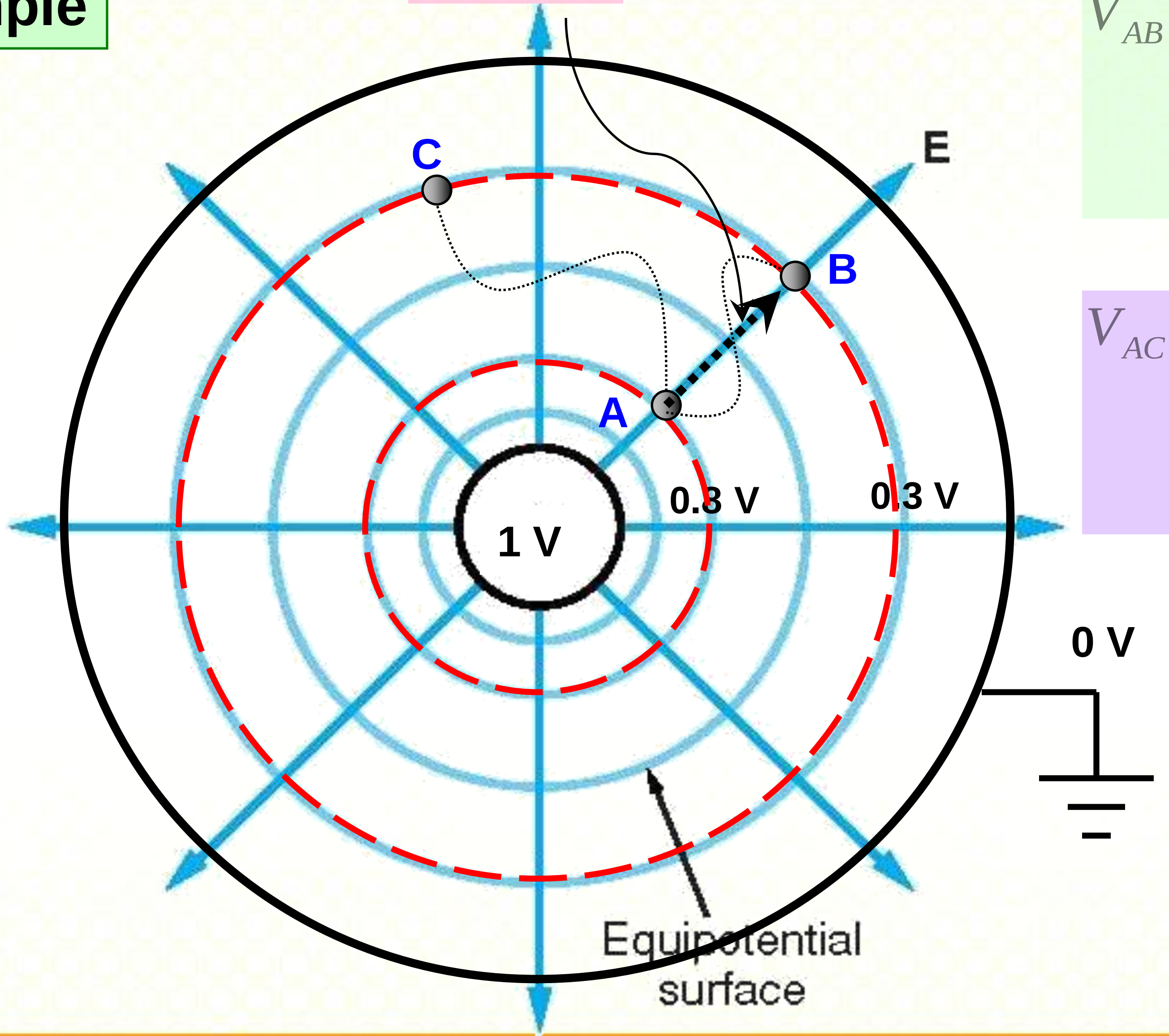
$$\vec{E} = -\vec{\nabla} V$$

The negative sign indicates that the field is pointing in the direction of decreasing potential.

Electric Potential (Gradient)

Example

$$\vec{E} = -\vec{\nabla} V$$



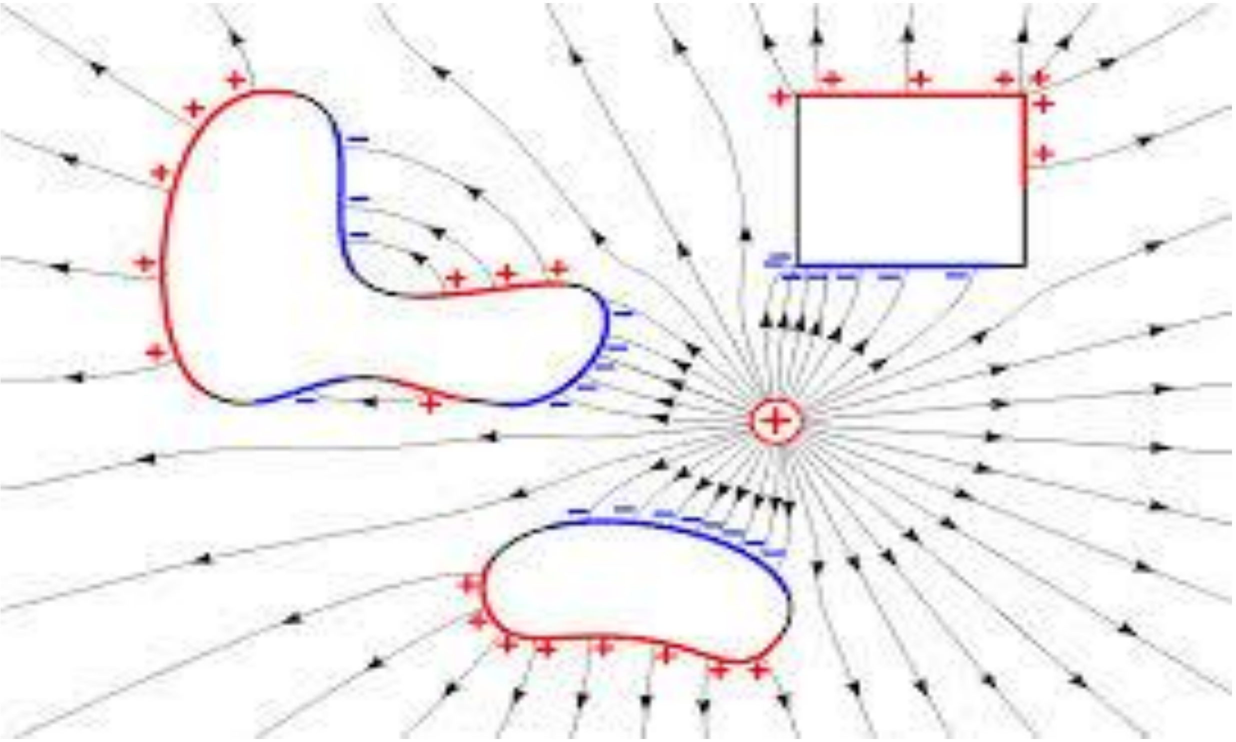
$$V_{AB} = V_B - V_A = (0.3 - 0.8)\,\text{V} = -0.5\ \text{V}$$

$$V_{AC} = V_C - V_A = (0.3 - 0.8)\,\text{V} = -0.5\ \text{V}$$

$$V_{AB} = V_{AC}$$

Electric Potential (Gradient)

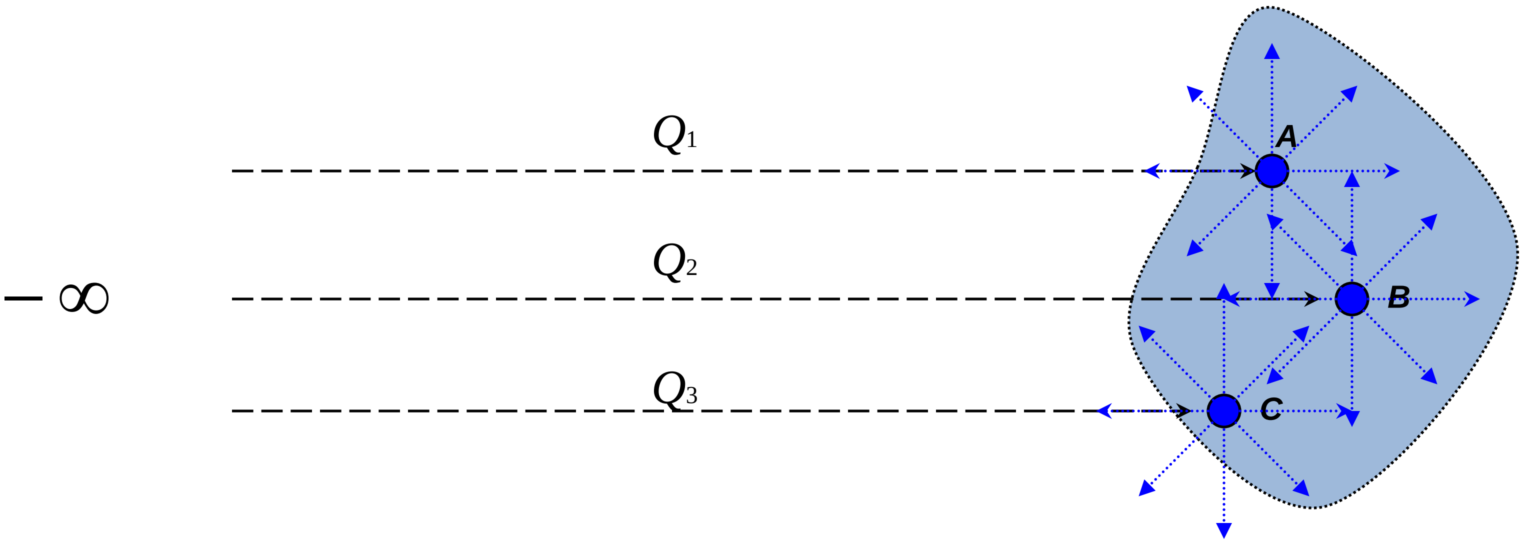
Example



2-D Equipotential Contours

Energy Density (Principle) (1)

1. The work required to bring charges from $r = -\infty$ into the defined space.

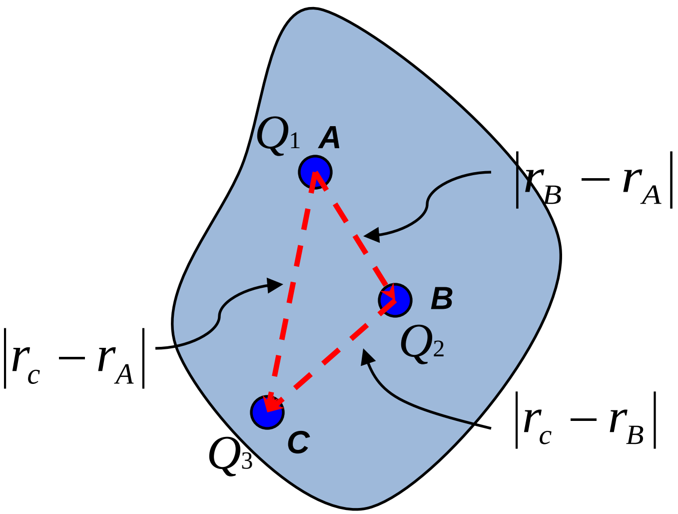


a) Moving the first charge, Q_1 requires no work since no force is required to move this charge in defined space

b) Moving the second charge, Q_2 requires work since the first charge, Q_1 creates an electric field.

c) Moving the third charge, Q_3 requires work since there are two charges already present.

Energy Density (Principle) (2)



Bring the first charge Q_1, the work done, W

$$W_1 = 0$$

Bring the second charge Q_2

$$W_2 = \int_{-\infty}^{r_B} \frac{Q_1 Q_2}{4\pi\varepsilon_o (r - r_A)^2} dr$$

$$= \frac{Q_1 Q_2}{4\pi\varepsilon_o |r_B - r_A|}$$

$$= Q_2 V_{12}$$

Bring the third charge Q_3

$$W_3 = -\int_{-\infty}^{r_C} \frac{Q_1 Q_3}{4\pi\varepsilon_o (r - r_A)^2} dr - \int_{-\infty}^{r_c} \frac{Q_2 Q_3}{4\pi\varepsilon_o (r - r_B)^2} dr$$

$$= \frac{Q_1 Q_3}{4\pi\varepsilon_o |r_C - r_A|} + \frac{Q_2 Q_3}{4\pi\varepsilon_o |r_C - r_B|}$$

$$= Q_3 V_{13} + Q_3 V_{23}$$

Energy Density (Principle) (3)

2. Hence the total work done in positioning the three charges is

$$\begin{aligned} W_{Total} &= W_1 + W_2 + W_3 \\ &= 0 + Q_2 V_{21} + Q_3 (V_{31} + V_{32}) \end{aligned} \quad (1)$$

3. If the charges were positioned in reverse order,

$$\begin{aligned} W_{Total} &= W_1 + W_2 + W_3 \\ &= Q_1 (V_{12} + V_{13}) + Q_2 V_{23} + 0 \end{aligned} \quad (2)$$

4. Adding (1) and (2) gives

$$\begin{aligned} 2\,W_{Total} &= Q_1 (V_{12} + V_{13}) + Q_2 (V_{21} + V_{23}) + Q_3 (V_{31} + V_{32}) \\ &= Q_1 V_{T1} + Q_2 V_{T2} + Q_3 V_{T3} \end{aligned}$$

Thus,

$$W_{Total} = \frac{1}{2} (Q_1 V_{T1} + Q_2 V_{T2} + Q_3 V_{T3})$$

where V_{T1} , V_{T2} and V_{T3} are total potential at *A*, *B* and *C*.

Energy Density (Principle) (4)

5. In general, if there are *n* point charges, *Q* the total work done, *W*

$$W_{Total} = \frac{1}{2}\sum_{i=1}^{n} Q_i V_{Ti}$$

or

$$W_{Total} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{Q_i Q_j}{4\pi\varepsilon_o \left|r_{ij}\right|} \qquad i \neq j$$

6. If the region has a continuous charge distribution, the total work done, *W*

$$W_{Total} = \frac{1}{2}\int \rho V \, dv$$

7. Since $\rho = \vec{\nabla} \cdot \vec{D}$

$$W_{Total} = \frac{1}{2}\int \left(\vec{\nabla} \cdot \vec{D}\right) V \, dv$$

Energy Density (Derivation) (1)

Example

Show $$\frac{1}{2}\int_v \rho V\,dv = \frac{1}{2}\int \varepsilon_o \vec{E}^2\,dv$$

Solution

Since $\rho = \vec{\nabla}\cdot\vec{D}$

$$\frac{1}{2}\int_v \rho V\,dv = \frac{1}{2}\int \left(\vec{\nabla}\cdot\vec{D}\right)V\,dv \qquad \textbf{(1)}$$

By using identity vector

$$\left(\vec{\nabla}\cdot\vec{D}\right)V = \vec{\nabla}\cdot V\vec{D} - \vec{D}\cdot\vec{\nabla}V$$

Thus, (1) becomes

$$\frac{1}{2}\int_v \rho V\,dv = \frac{1}{2}\int_v \left(\vec{\nabla}\cdot V\vec{D}\right)dv - \frac{1}{2}\int_v \left(\vec{D}\cdot\vec{\nabla}V\right)dv \qquad \textbf{(2)}$$

Energy Density (Derivation) (2)

By using divergence theorem

$$\frac{1}{2}\int_v \rho V\, dv = \frac{1}{2}\oint_S \left(V\vec{D}\right)\cdot d\vec{S} - \frac{1}{2}\int_v \left(\vec{D}\cdot\vec{\nabla}V\right)dv$$

We know that $V \propto 1/r$ **and** $\vec{D} \propto 1/r^2$ **thus,** $V\vec{D} \propto 1/r^3$

For surface area $dS \propto r^2$ **that**

$$\left(V\vec{D}\right)\cdot dS \propto \frac{1}{r}$$

When surface, S **become large,** $1/r = 0$ **thus, (2) reduces to**

$$\frac{1}{2}\int_v \rho V\, dv = -\frac{1}{2}\int_v \left(\vec{D}\cdot\vec{\nabla}V\right)dv$$

$$= \frac{1}{2}\int_v \left(\vec{D}\cdot\vec{E}\right)dv$$

$$= \frac{1}{2}\int_v \varepsilon_o E^2\, dv$$

Laplace's Equation & Poisson's Equation (1)

1. For fundamental electrostatics,

$$\vec{\nabla}\cdot\vec{D}=\rho_v \qquad (1) \qquad\qquad \vec{\nabla}\times\vec{E}=0 \qquad (2)$$

2. From (1), electrostatic field in linear, isotropic medium,

$$\vec{\nabla}\cdot\vec{D}=\vec{\nabla}\cdot\varepsilon\vec{E}$$
$$=\rho_v$$

3. From (2), electrostatic field is irrotational and $\vec{E}=-\vec{\nabla}V$

$$\vec{\nabla}\cdot\varepsilon\,\vec{E}=\vec{\nabla}\cdot\left(-\,\varepsilon\vec{\nabla}V\right)$$
$$=\rho_v$$

4. In homogeneous medium, ε is constant

$$\vec{\nabla}\cdot\left(\vec{\nabla}V\right)=\nabla^2 V$$
$$=-\frac{\rho_v}{\varepsilon}$$

Poisson's equation

Laplace's Equation & Poisson's Equation (2)

5. In a charge-free medium (in air), $\rho_v = 0$

$$\nabla^2 V = 0$$

Laplace's equation

6. ∇^2 is called as Laplacian operator.

7. Laplace's equation is important to solve scalar electrostatic problems involving a set of conductors maintained at different potentials. (include capacitor)

8. Laplace's equation can be in Cartesian, cylindrical, or spherical coordinates.

Laplace's Equation & Poisson's Equation (3)

Cartesian coordinate

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0 \qquad V(x,y,z)$$

Cylindrical coordinate

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial z^2}=0 \qquad V(\rho,\phi,z)$$

Spherical coordinate

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}=0 \qquad V(r,\theta,\phi)$$

V **must is a scalar function**

Uniqueness Theorem (1)

1. Solving the electromagnetic differential equations in a domain of interest $\left(\vec{E},\ \vec{H},\ \vec{D},\ B \text{ and } \vec{J}\right)$, one may obtain many solutions.

2. However, there is only one real solution to the problem.

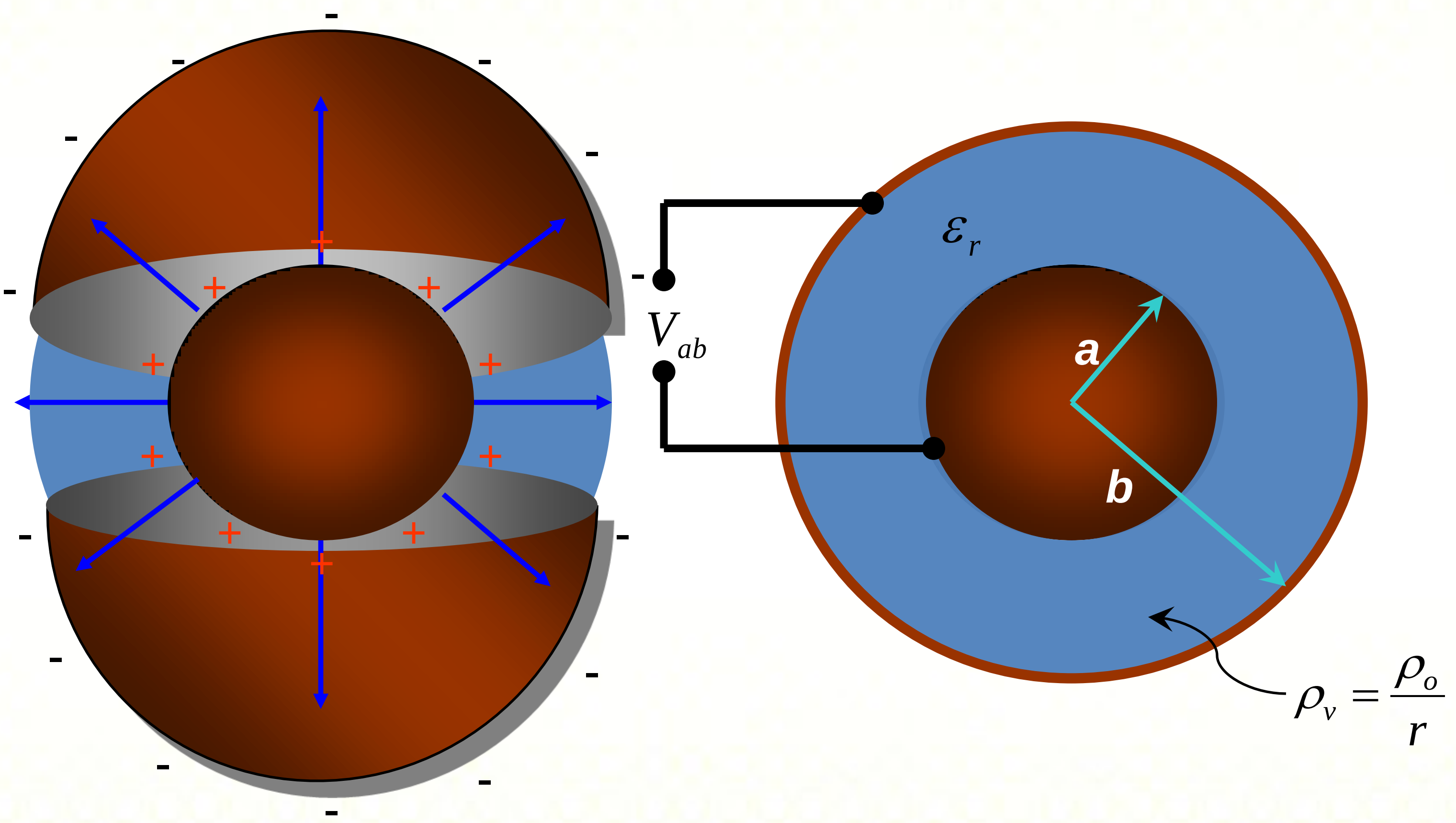
3. To find this real solution, one should know the boundary conditions associated with the domain.

4. Thus, any solution differential equation (Laplace' s equation) that satisfies the boundary conditions must be the only solution regardless of the methods used.

5. This is known as the uniqueness theorem.

Scalar Electrostatic Problems (1)

Determine the difference potential, V_{ab} in the dielectric region between a pair of concentric spheres.



Scalar Electrostatic Problems (2)

Solution

Using Poisson's equation,

$$\nabla^2 V = -\frac{\rho_V}{\varepsilon} = -\frac{\rho_0}{r\varepsilon} \qquad (1)$$

The potential, V is only a function of radial, r in spherical coordinate

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V(r)}{\partial r}\right) = -\frac{\rho_0}{r\varepsilon_r\varepsilon_0} \qquad (2)$$

Integrating (2) respect to dr for both side

$$r^2 \frac{\partial V(r)}{\partial r} = -\frac{\rho_0 r^2}{2\varepsilon_r\varepsilon_0} + A$$

Integrating second time respect to dr for both side

$$V(r) = -\frac{\rho_0 r}{2\varepsilon_r\varepsilon_0} - \frac{A}{r} + B$$

Scalar Electrostatic Problems (3)

The absolute potential or equipotential surface for inner conductor and outer conductor

$$V(r=a)=V_a=-\frac{\rho_0 a}{2\varepsilon_r\varepsilon_0}-\frac{A}{a}+B$$

$$V(r=b)=V_b=-\frac{\rho_0 b}{2\varepsilon_r\varepsilon_0}-\frac{A}{b}+B=0$$

Outer conductor is grounded

The difference potential, V_{ab} **between inner conductor and outer conductor**

$$V_{ab}=V_a-V_b=\left(-\frac{\rho_0 a}{2\varepsilon_r\varepsilon_0}-\frac{A}{a}+B\right)-\left(-\frac{\rho_0 b}{2\varepsilon_r\varepsilon_0}-\frac{A}{b}+B\right) \quad (3)$$

$$V_a=\frac{\rho_0}{2\varepsilon_r\varepsilon_0}(b-a)-\left(A\frac{(b-a)}{ab}\right)$$

From (3),

$$A=\frac{V_a ab}{a-b}+\frac{\rho_0 ab}{2\varepsilon_r\varepsilon_0} \qquad B=\frac{\rho_0(a+b)}{2\varepsilon_r\varepsilon_0}+\frac{V_a ab}{a-b}$$

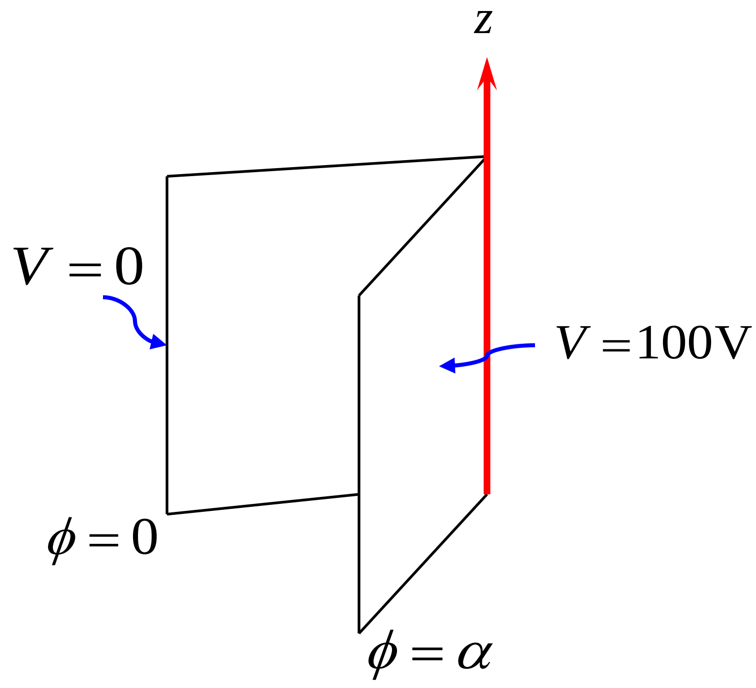
Scalar Electrostatic Problems (4)

Finally, the difference potential, V_{ab} between the spheres,

$$V(r) = \frac{\rho_o}{2\varepsilon_r\varepsilon_o}\left(\frac{a}{r} - r\right) + \frac{V_a a}{r}$$

Scalar Electrostatic Problems (5)

Two infinite radial conductor planes with an interior angle, α as shown in Figure.



The **potential** and **electric field density** in the region between the conductors by utilizing the uniqueness theorem.

Scalar Electrostatic Problems (6)

Solution

Laplace's equation

$$\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

The solution is

$$V = A\phi + B$$

The condition

$$0 = A(0) + B \qquad \textbf{and} \qquad 100 = A(\alpha) + B$$

Thus,

$$A = \frac{100}{\alpha} \qquad \textbf{and} \qquad B = 0$$

Scalar Electrostatic Problems (7)

The potential in the region between the conductors

$$V = 100\frac{\phi}{\alpha}\ \text{V}$$

The electric field in the region between the conductors

$$\begin{aligned}\vec{E} &= -\vec{\nabla}V \\ &= -\frac{1}{\rho}\frac{\partial}{\partial\phi}\left(100\frac{\phi}{\alpha}\right)\hat{\phi} \\ &= -\frac{100}{\rho\alpha}\hat{\phi}\ (\text{V/m})\end{aligned}$$

Scalar Electrostatic Problems (8)

Further Solution

The **flux density,** $\vec{D}$ **in the region between the conductors. For air,** $\varepsilon_r = 1$

$$\vec{D} = \varepsilon_o \varepsilon_r \vec{E}$$
$$= -\frac{100\varepsilon_o}{\rho\alpha}\hat{\phi} \quad (\mathrm{V/m})$$

The **surface charge density,** ρ_s

$$\rho_s = D_n$$
$$= -D_\phi$$
$$= \frac{100\varepsilon_o}{\rho\alpha} \left(\mathrm{C/m^2}\right)$$

Scalar Electrostatic Problems (9)

Further Solution

The total charge, Q at the conductor plate

$$Q = \int_S \rho_s \, dS$$

$$= \int_{z_1}^{z_2} \int_{\rho_1}^{\rho_2} \frac{100\varepsilon_o}{\rho\alpha} \, d\rho \, dz$$

$$= \frac{100\varepsilon_o (z_2 - z_1)}{\alpha} \ln \frac{\rho_2}{\rho_1}$$

The capacitance, C between conductor plate

$$C = \frac{Q}{V_o}$$

$$= \frac{\varepsilon_o (z_2 - z_1)}{\alpha} \ln \frac{\rho_2}{\rho_1}$$

What is the resistance, R between conductor plate ?

References

J. A. Edminister. *Schaum's outline of Theory and problems of electromagnetics,* 2nd Ed. New York: McGraw-Hill. 1993.

M. N. O. Sadiku. *Elements of Electromagnetics,* 3th Ed. U.K: Oxford University Press. 2010.

F. T. Ulaby. *Fundamentals of Applied Electromagnetics,* Media ed. New Jersey: Prentice Hall. 2001.

Bhag Singh Guru and Hüseyin R. Hiziroglu. *Electromagnetic Field Theory Fundamentals*, 2nd Ed. U.K.: Cambridge University Press. 2009.

W. H. Hayt. Jr. *Engineering Electromagnetics*, 5th Ed. New York: McGraw-Hill. 2009.