SAB2223 Mechanics of Materials and Structures

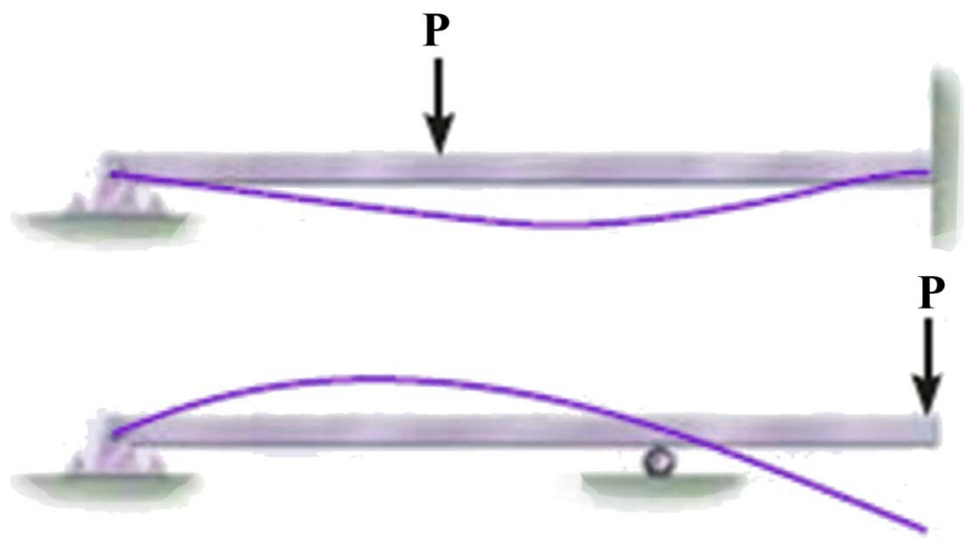
TOPIC 7
DEFLECTIONS OF BEAMS

Lecturer:
Dr. Shek Poi Ngian

TOPIC 7
DEFLECTIONS OF BEAMS

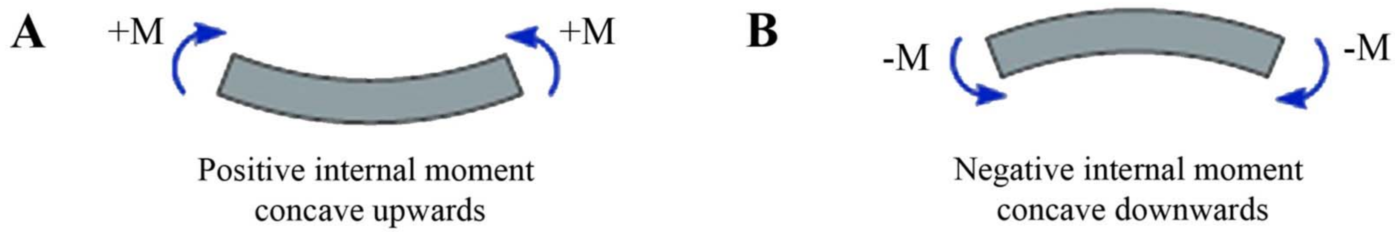
Elastic Curve

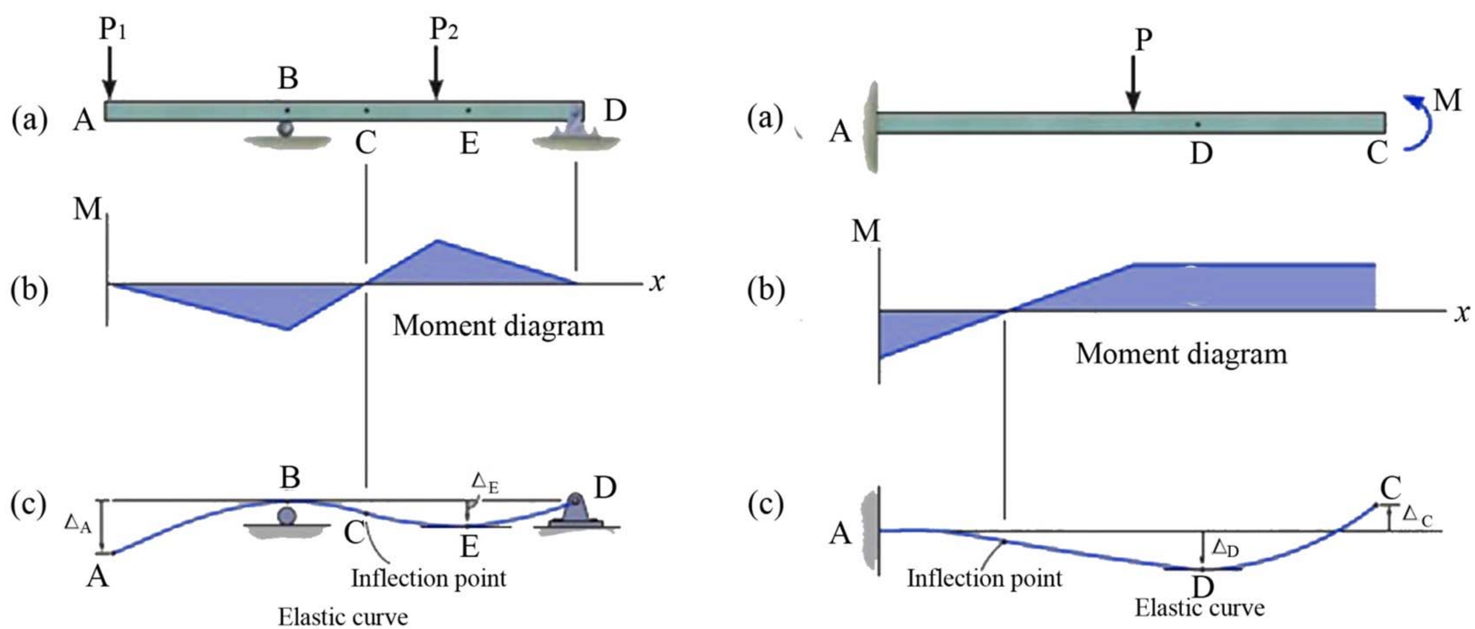
- The deflection diagram of the longitudinal axis that passes through the centroid of each cross-sectional area of the beam is called the elastic curve, which is characterized by the deflection and slope along the curve



Elastic Curve

- Moment-curvature relationship:
  - Sign convention:





Elastic Curve

ε = (ds' – ds)/ds

ds = dx = ρdθ

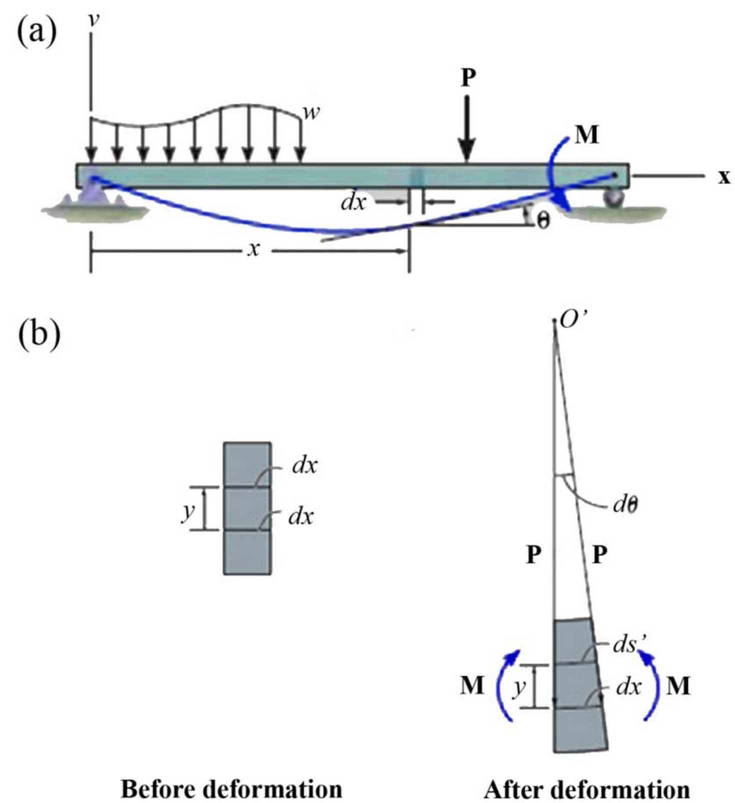
ds' = (ρ-y) dθ

ε = [(ρ – y) dθ – ρdθ] / (ρdθ)

$$\frac{1}{\rho} = -\frac{\varepsilon}{y}$$

ε = σ / E and σ = -My/I

$$\frac{1}{\rho} = \frac{M}{EI} \text{ or } \frac{1}{\rho} = -\frac{\sigma}{E_y}$$



Slope and Displacement by Integration

Kinematic relationship:

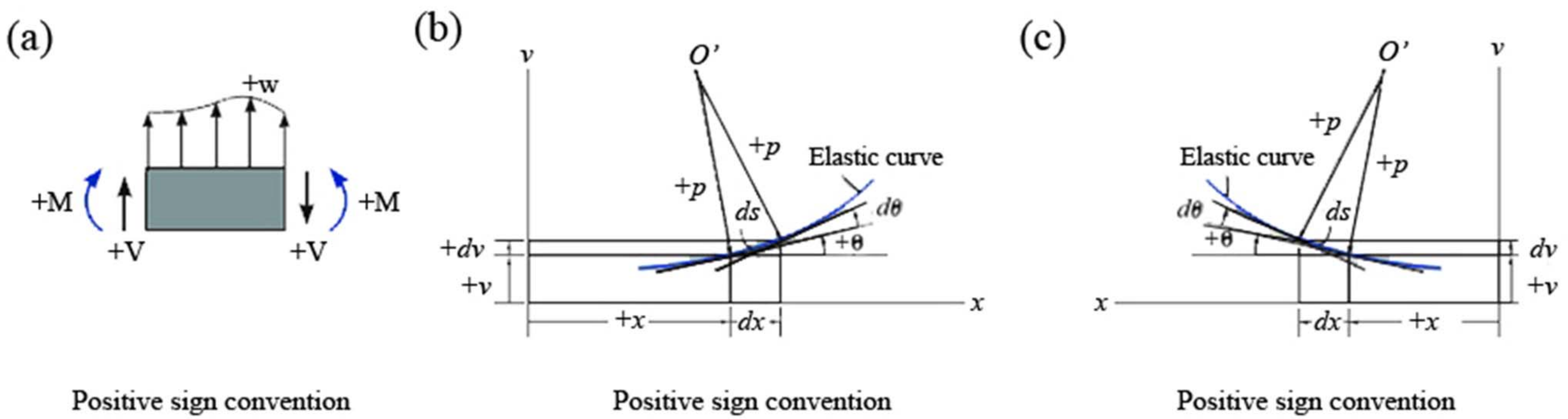
$$\frac{1}{\rho} = -\frac{d^2v/dv^2}{\left[1+\left(dv/dx\right)^2\right]^{3/2}}$$

Moment curvature equation:

$$\frac{M}{EI} = \frac{1}{\rho} = \frac{d^2v/dx^2}{\left[1+\left(dv/dx\right)^2\right]^{3/2}} \approx \frac{d^2v}{dx^2}$$

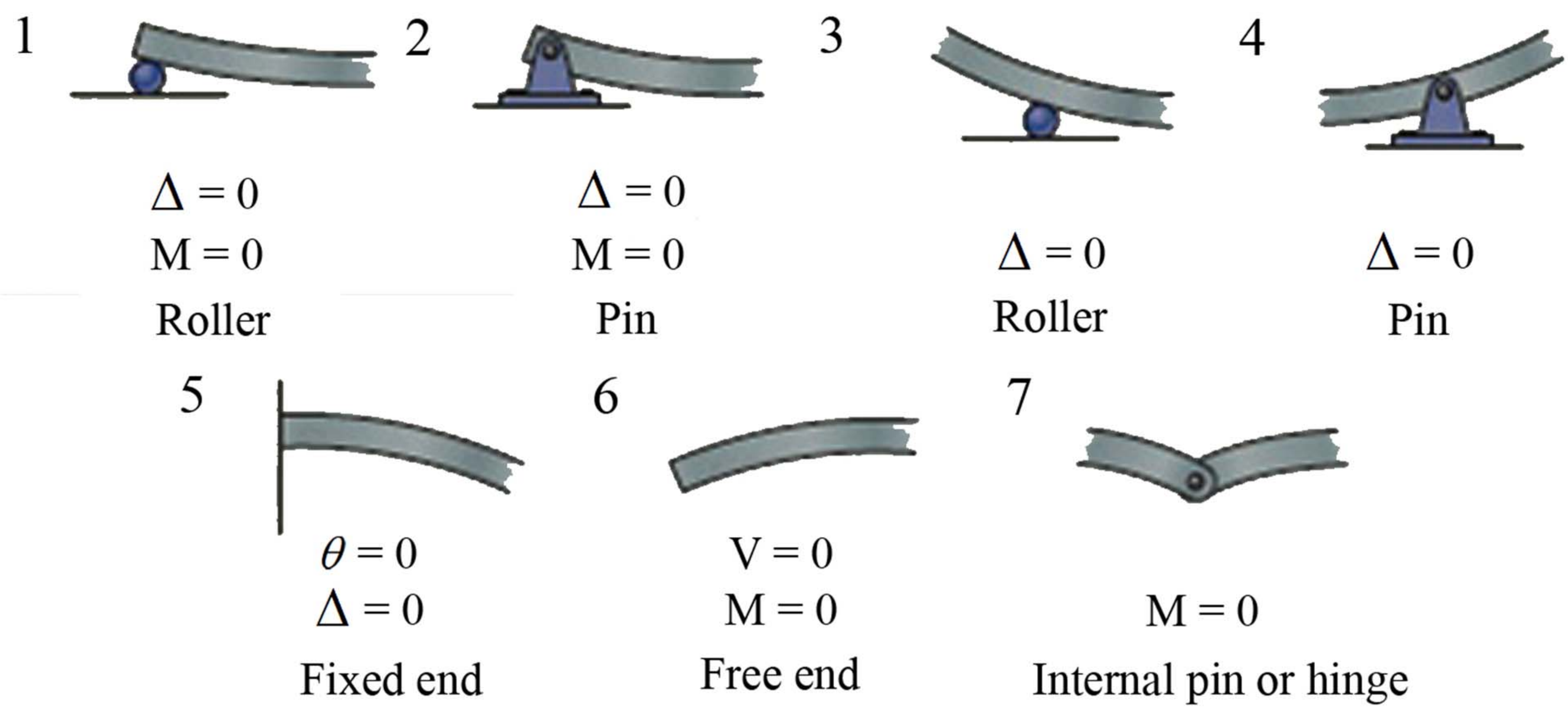
Slope and Displacement by Integration

- Sign convention:



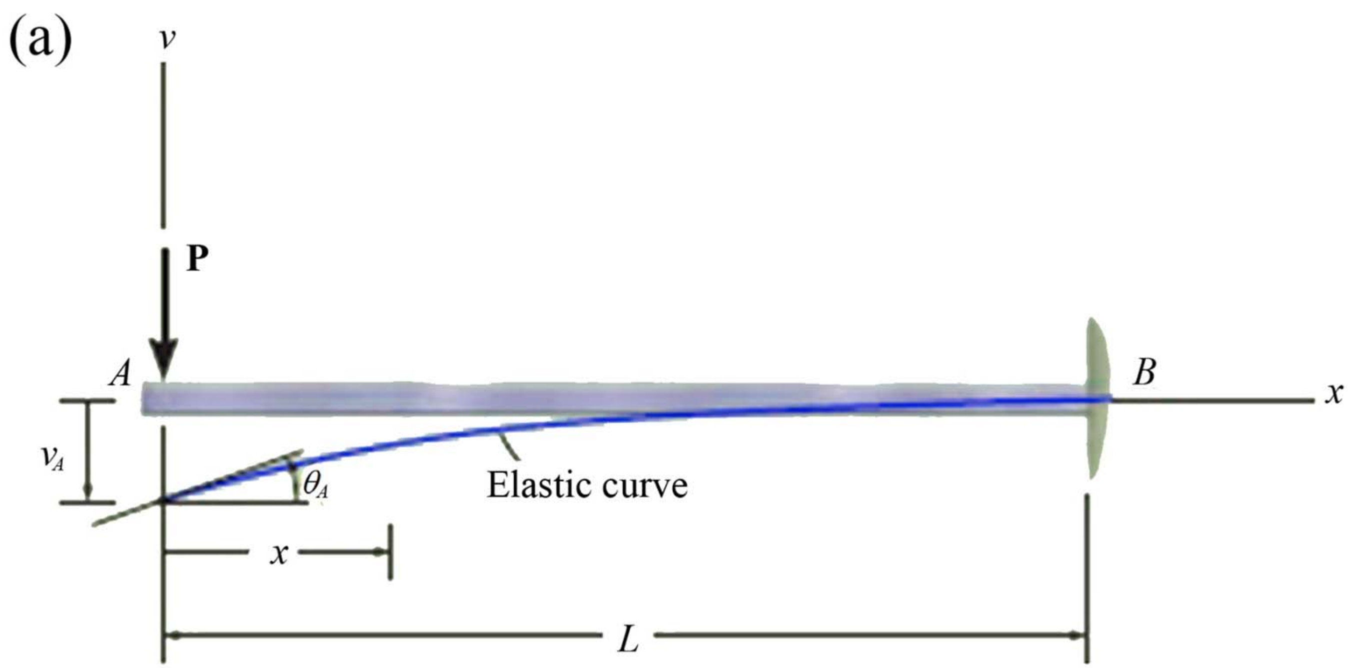
Slope and Displacement by Integration

- Boundary Conditions:
  - The integration constants can be determined by imposing the boundary conditions, or
  - Continuity condition at specific locations



Example 1

The cantilevered beam shown below is subjected to a vertical load **P** at its end. Determine the equation of the elastic curve. *EI* is constant.



Example 1 (cont.)

Solutions

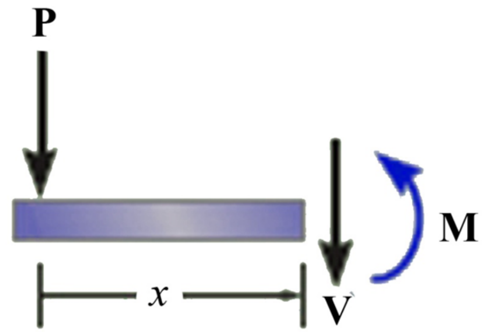
$$M = -Px$$

$$EI\frac{d^2v}{dx^2} = -Px \qquad (1)$$

$$EI\frac{dv}{dx} = -\frac{Px^2}{2} + C_1 \qquad (2)$$

$$EIv = -\frac{Px^3}{6} + C_1x + C_2 \qquad (1)$$

(b)



Example 1 (cont.)

Solutions

- dv/dx = 0 at x = L and v = 0 at x = L

$$0 = -\frac{PL^2}{2} + C_1$$

$$0 = -\frac{PL^3}{6} + C_1 L + C_2$$

$$\Rightarrow C_1 = \frac{PL^2}{2} \text{ and } C_2 = -\frac{PL^3}{3}$$

- Substituting C1 and C2

$$\theta = \frac{P}{2EI}\left(L^2 - x^2\right)$$

$$v = \frac{P}{6EI}\left(-x^3 + 3L^2 x - 2L^3\right) \quad \text{(Ans)}$$

Example 1 (cont.)

Solutions

- Maximum slope and displacement occur at A(x =0),

$$\theta_A = \frac{PL^2}{2EI} \quad (4)$$

$$v_A = -\frac{PL^3}{3EI} \quad (5)$$

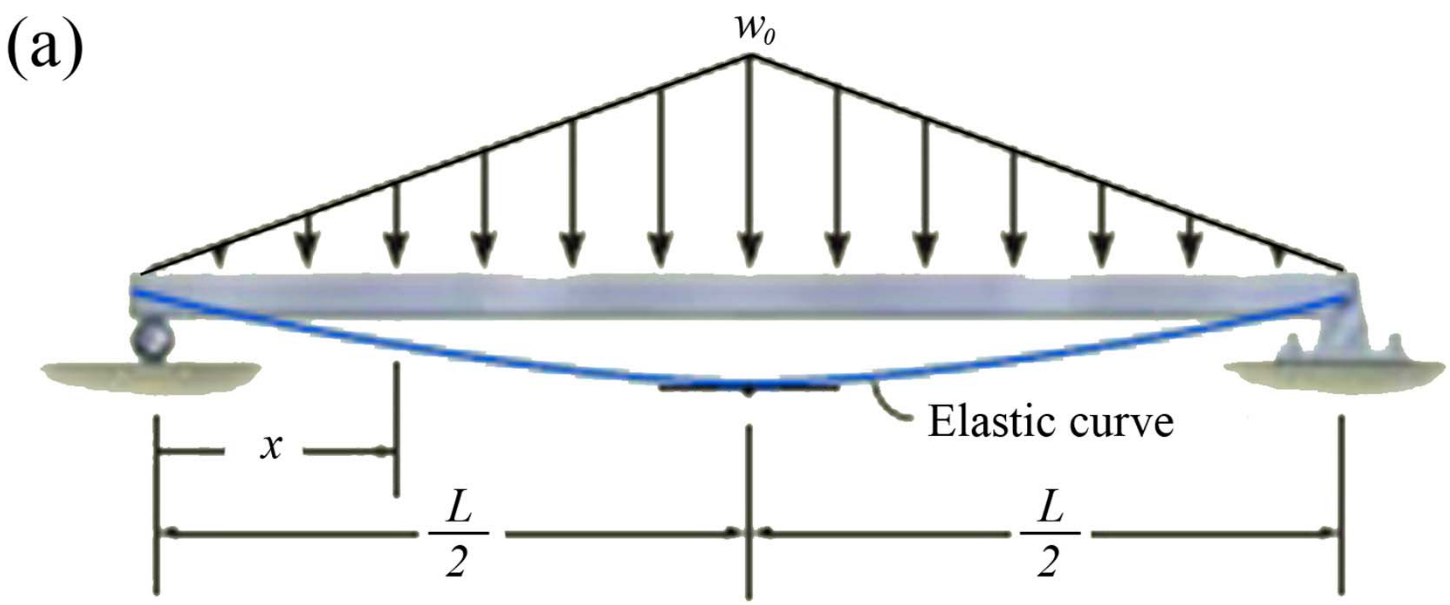
- yield stress is 250 MPa; E = 200 kN/mm^2; I = 84.4 × 10^6 mm^4

$$\theta_A = \frac{30(5)^2(1000)^2}{2[200][84.4(10^6)]} = 0.0222 \text{ rad}$$

$$v_A = -\frac{30(5)^2(1000)^2}{3[200][84.4(10^6)]} = -74.1 \text{ mm}$$

Example 2

The simply supported beam shown below supports the triangular distributed loading. Determine its maximum deflection. *EI* is constant.



Example 2 (cont.)

Solutions

$$0 \le x \le L/2$$

$$w = \frac{2w_0}{L}x$$

$$+\sum M_{NA} = 0; \qquad M + \frac{w_0 x^2}{L}\left(\frac{x}{3}\right) - \frac{w_0 L}{4}(x) = 0$$

$$M = -\frac{w_0 x^2}{3L} + \frac{w_0 L}{4}x$$

Example 2 (cont.)

Solutions

$$EI\frac{d^2v}{dx^2} = M = -\frac{w_0}{3L}x^3 + \frac{w_0L}{4}x$$

$$EI\frac{dv}{dx} = -\frac{w_0}{12L}x^4 + \frac{w_0L}{8}x^2 + C_1$$

$$EIv = -\frac{w_0}{60L}x^5 + \frac{w_0L}{24}x^3 + C_1x + C_2$$

$v = 0, x = 0$ and $dv/dx = 0, x = L/2$

$$C_1 = -\frac{5w_0L^3}{192}, C_2 = 0$$

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Example 2 (cont.)

Solutions

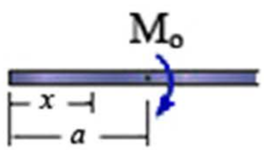
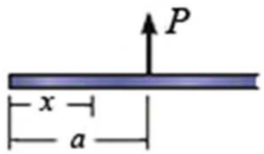
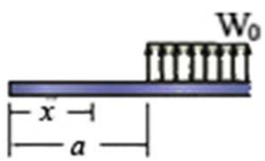
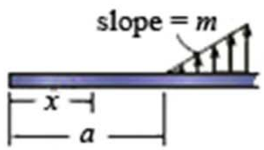
$$EIv = -\frac{w_0}{60L}x^5 + \frac{w_0 L}{24}x^3 - \frac{5w_0 L^3}{192}x$$

- Maximum deflection at x = L/2,

$$v_{\max} = -\frac{w_0 L^4}{120EI}$$

Use of Continuous Functions

Macaulay functions

Loading	Loading Function $w = w(x)$	Shear $V = \int w(x)dx$	*Moment* $M = \int V\,dx$
1. 	$w = M_o(x-a)^{-2}$	$v = M_o(x-a)^{-1}$	$M = M_o(x-a)^0$
2. 	$w = P\,(x-a)^{-1}$	$v = P\,(x-a)^0$	$M = P\,(x-a)^1$
3. 	$w = W_0\,(x-a)^0$	$v = W_0\,(x-a)^1$	$M = \frac{W_0}{2}(x-a)^2$
4. 	$w = m\,(x-a)^1$	$v = \frac{m}{2}(x-a)^2$	$M = \frac{m}{6}(x-a)^3$

Use of Continuous Functions

- *Macaulay* functions

$$\langle x-a\rangle^n = \begin{cases} 0 & \text{for } x < a \\ (x-a)^n & \text{for } x \geq a \end{cases}$$

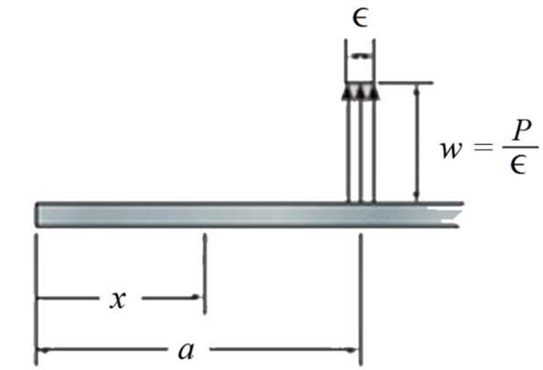
$$n \geq a$$

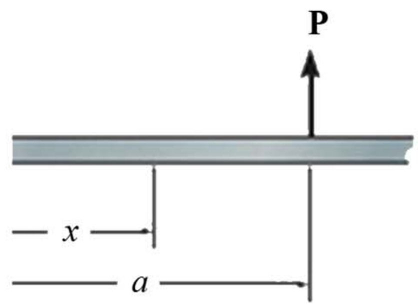
- Integration of *Macaulay* functions:

$$\int \langle x-a\rangle^n dx = \frac{\langle x-a\rangle^{n+1}}{n+1} + C$$

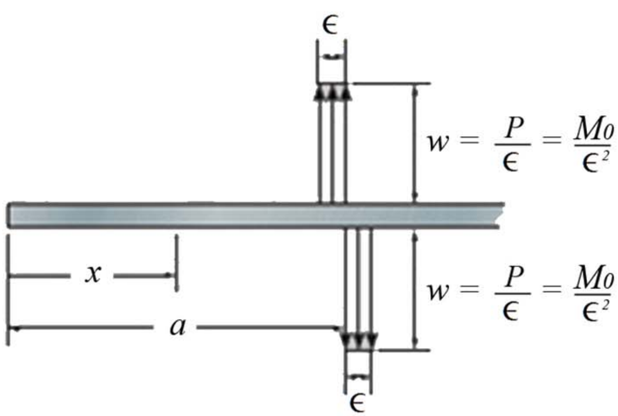
Use of Continuous Functions

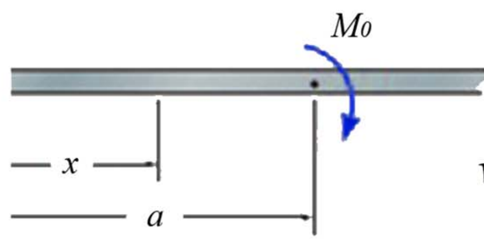
- Singularity Functions:





$$w = P\langle x-a\rangle^{-1} = \begin{cases} 0 & \text{for } x \neq a \\ P & \text{for } x = a \end{cases}$$

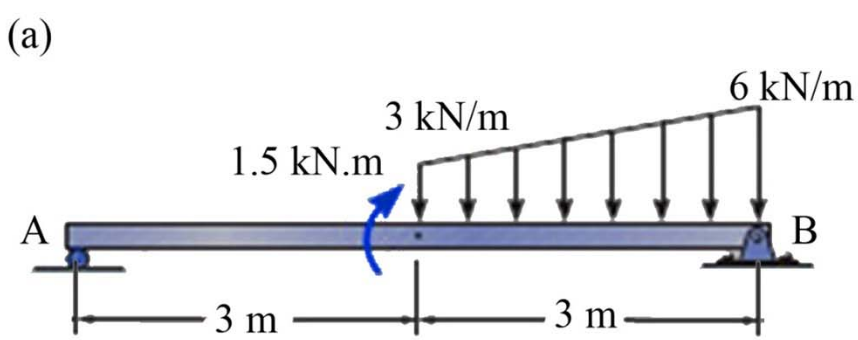


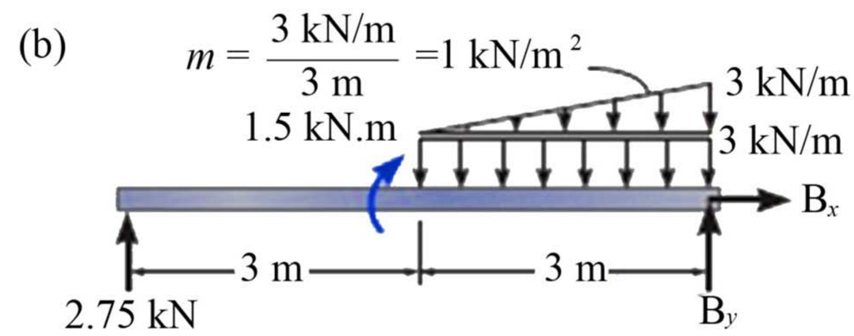


$$w = M_0\langle x-a\rangle^{-2} = \begin{cases} 0 & \text{for } x \neq a \\ M_0 & \text{for } x = a \end{cases}$$

Use of Continuous Functions

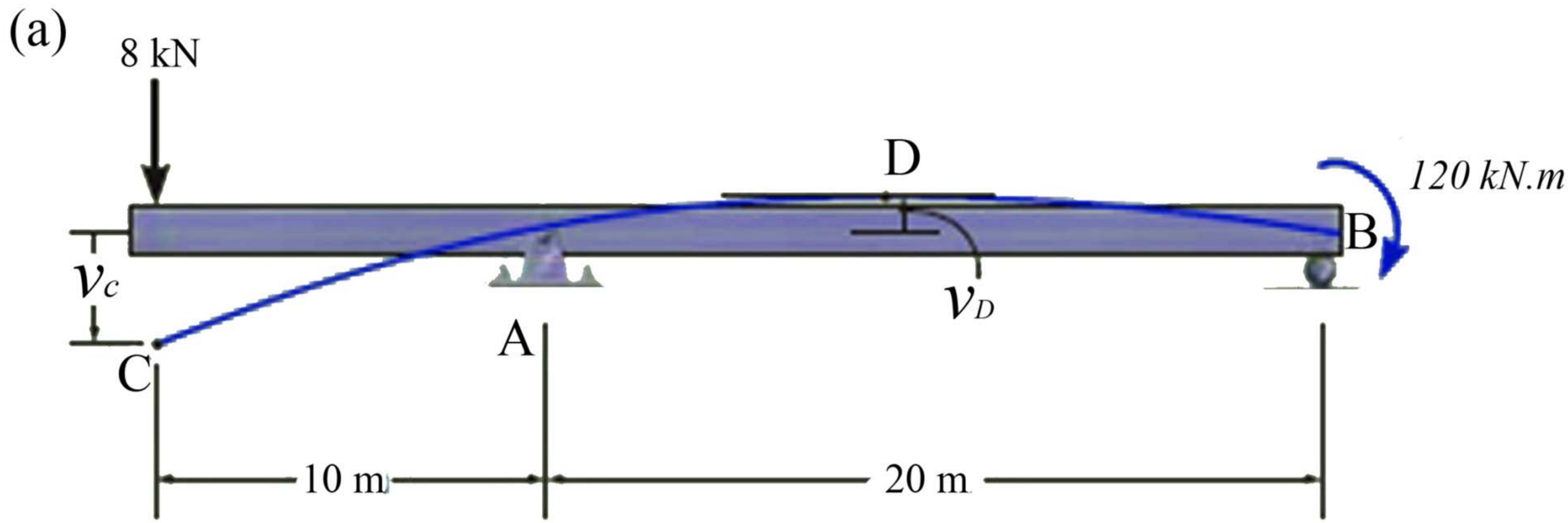
$$\int \langle x-a \rangle^{n} dx = \langle x-a \rangle^{n+1}, n = -1, -2$$





Example 3

Determine the maximum deflection of the beam shown below. *EI* is constant.



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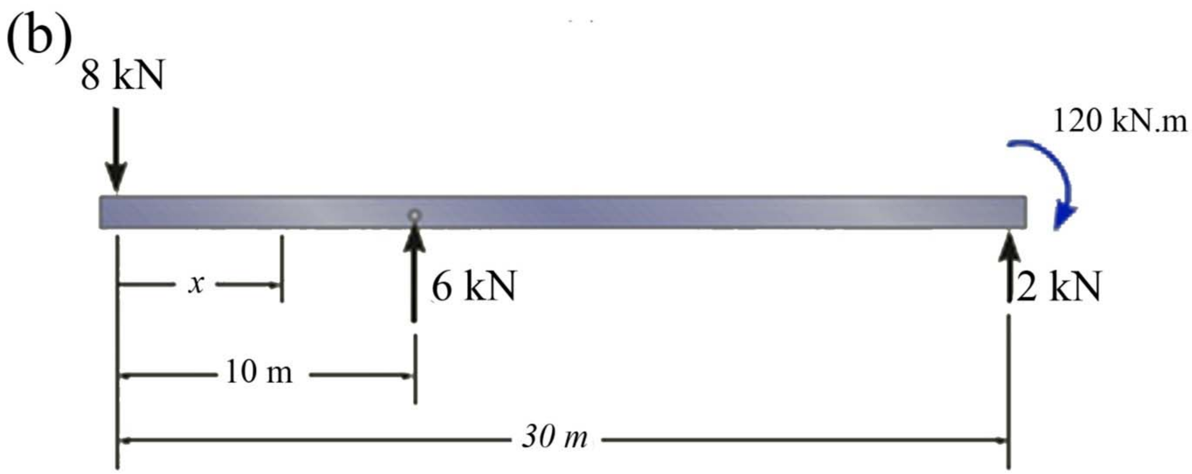
Example 3 (cont.)

Solutions



$$w = -8\langle x-0\rangle^{-1} + 6\langle x-10\rangle^{-1}$$

Example 3 (cont.)

Solutions

$$V = -8\langle x-0\rangle^{0} + 6\langle x-10\rangle^{0}$$

$$\begin{aligned} M &= -8\langle x-0\rangle^{1} + 6\langle x-10\rangle^{1} \\ &= \left(-8x + 6\langle x-10\rangle^{1}\right)\mathrm{kN\cdot m} \end{aligned}$$

$$EI\frac{d^2v}{dx^2} = -8x + \langle x-10\rangle^{1}$$

$$EI\frac{dv}{dx} = -4x^2 + 3\langle x-10\rangle^{2} + C_1$$

$$EIv = -\frac{4}{3}x^3 + \langle x-10\rangle^{3} + C_1x + C_2 \quad (1)$$

Example 3 (cont.)

Solutions

- v = 0 at x = 10 m and at x = 30 m,

$$0 = -1333 + (10 - 10)^3 + C_1(10) + C_2$$

$$0 = -36000 + (30 - 10)^3 + C_1(30) + C_2$$

$$\Rightarrow C_1 = 1333 \text{ and } C_2 = -12000$$

$$EI\frac{dv}{dx} = -4x^2 + 3\langle x - 10\rangle^2 + 1333 \quad (2)$$

$$EIv = -\frac{4}{3}x^3 + \langle x - 10\rangle^3 + 1333x - 12000 \quad (3)$$

Example 3 (cont.)

Solutions

$$v_C = -\frac{12000}{EI}\ \text{kN}\cdot\text{m}^3$$

$$0 = -x_D^2 + 3\langle x_D - 10\rangle^2 + 1333$$

$$x_D^2 + 60x_D - 1633 = 0$$

Solving for the positive root, $x_D = 20.3\ \text{m}$

Example 3 (cont.)

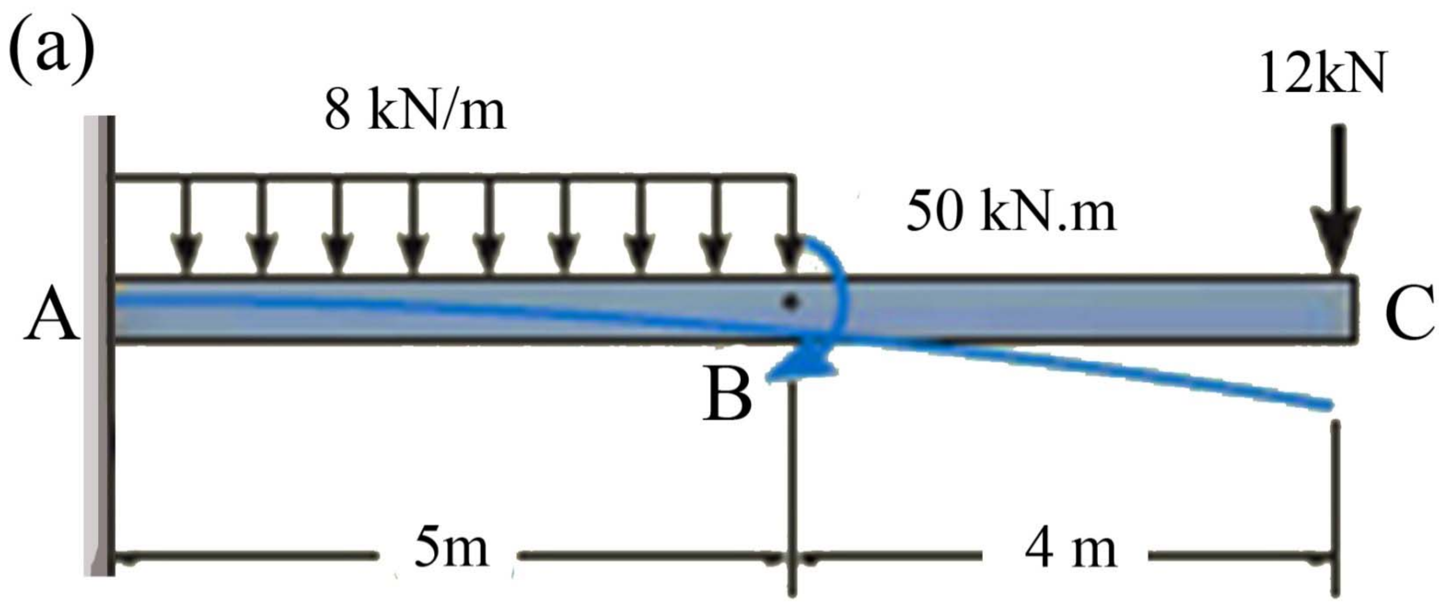
Solutions

$$EIv_D = -\frac{4}{3}(20.3)^3 + (20.3-10)^3 + 1333(20.3) - 12000$$

$$v_D = \frac{5006}{EI}\,\mathrm{kN\cdot m^3}$$

Example 4

Determine the equation of the elastic curve for the cantilevered beam shown below. *EI* is constant.



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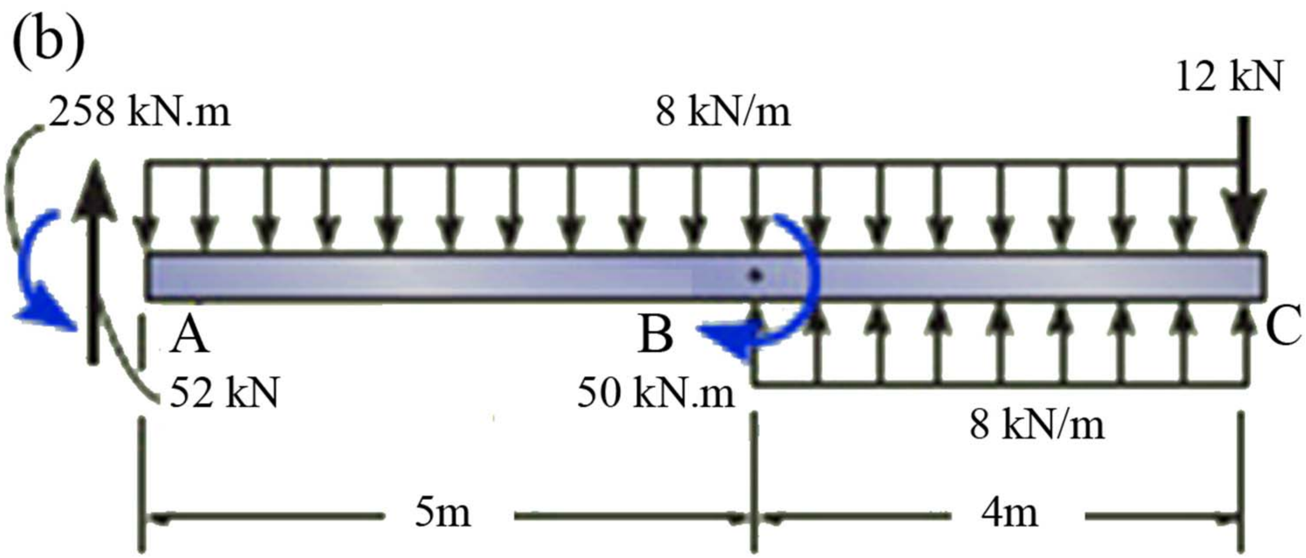
Example 4 (cont.)

Solutions



$$w = -52\langle x-0\rangle^{-1} + 258\langle x-0\rangle^{-2} + \langle x-0\rangle^{-0} - 50\langle x-5\rangle^{-2} - 8\langle x-5\rangle^{0}$$

UTM

Example 4 (cont.)

Solutions

$$dV/dx = -w(x) \text{ and } dM/dx = V$$

$$V = 52\langle x-0\rangle^0 - 258\langle x-0\rangle^{-1} - 8\langle x-0\rangle^1 + 50\langle x-5\rangle^{-1} + 8\langle x-5\rangle^1$$

$$M = -258\langle x-0\rangle^0 + 52\langle x-0\rangle^1 - \frac{1}{2}(8)\langle x-0\rangle^2 + 50\langle x-5\rangle^0 + \frac{1}{2}(8)\langle x-5\rangle^2$$

$$= \left(-258 + 52x - 4x^2 + 50\langle x-5\rangle^0\right) + 4\langle x-5\rangle^2 \text{ kN}\cdot\text{m}$$

$$EI\frac{d^2v}{dx^2} = -258 + 52x - 4x^2 + 50\langle x-5\rangle^0 + 4\langle x-5\rangle^2$$

$$EI\frac{dv}{dx} = -258x + 26x^2 - \frac{4}{3}x^3 + 50\langle x-5\rangle^1 + \frac{4}{3}\langle x-5\rangle^3 + C_1$$

$$EIv = -129x^2 + \frac{26}{3}x^3 - \frac{1}{3}x^4 + 25\langle x-5\rangle^2 + \frac{1}{3}\langle x-5\rangle^4 + C_1x + C_2$$

Example 4 (cont.)

Solutions

- dv/dx = 0, x = 0, C1 = 0; and v = 0, C2 = 0

$$v = \frac{1}{EI}\left(-129x^2 + \frac{26}{3}x^3 - \frac{1}{3}x^4 + 25\langle x-5\rangle^2 + \frac{1}{3}\langle x-5\rangle^4 \right) \text{ m}$$

Moment Area Method

Theorem 1:

$$EI\frac{d^2v}{dx^2} = EI\frac{d}{dx}\left(\frac{dy}{dx}\right) = M$$

- θ ≈ dv/dx, so

$$d\theta = \left(\frac{M}{EI}\right)dx$$

- Therefore,

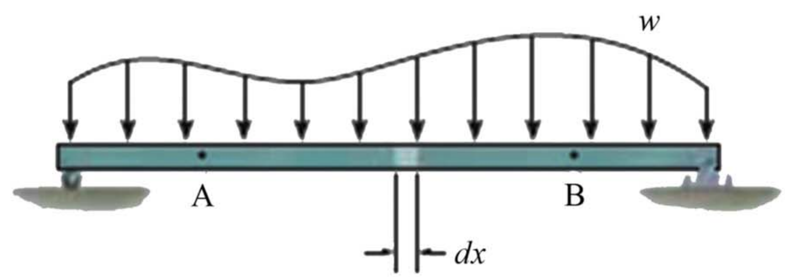
$$\theta_{B/A} = \int_A^B \frac{M}{EI}dx$$

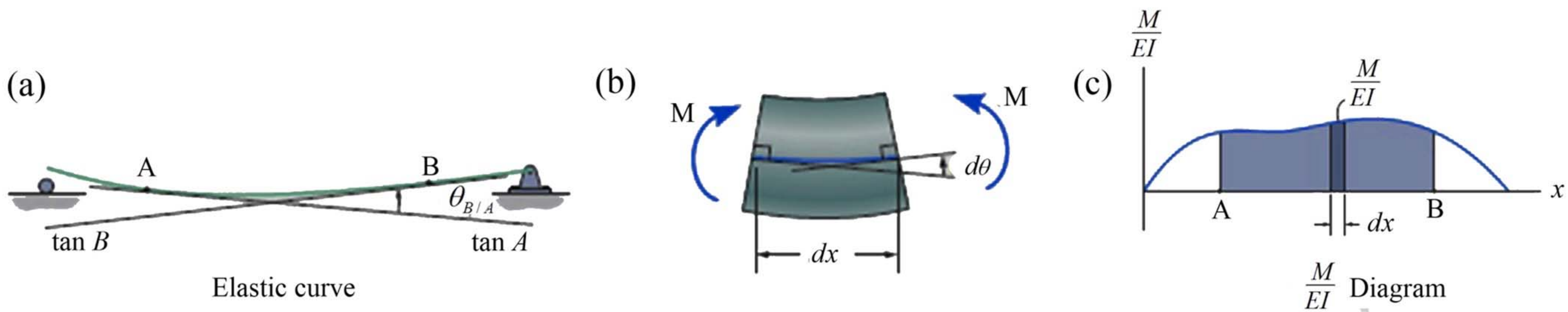
Moment Area Method

Theorem 1 (cont.):

- This equation forms the basis for the first moment-area theorem

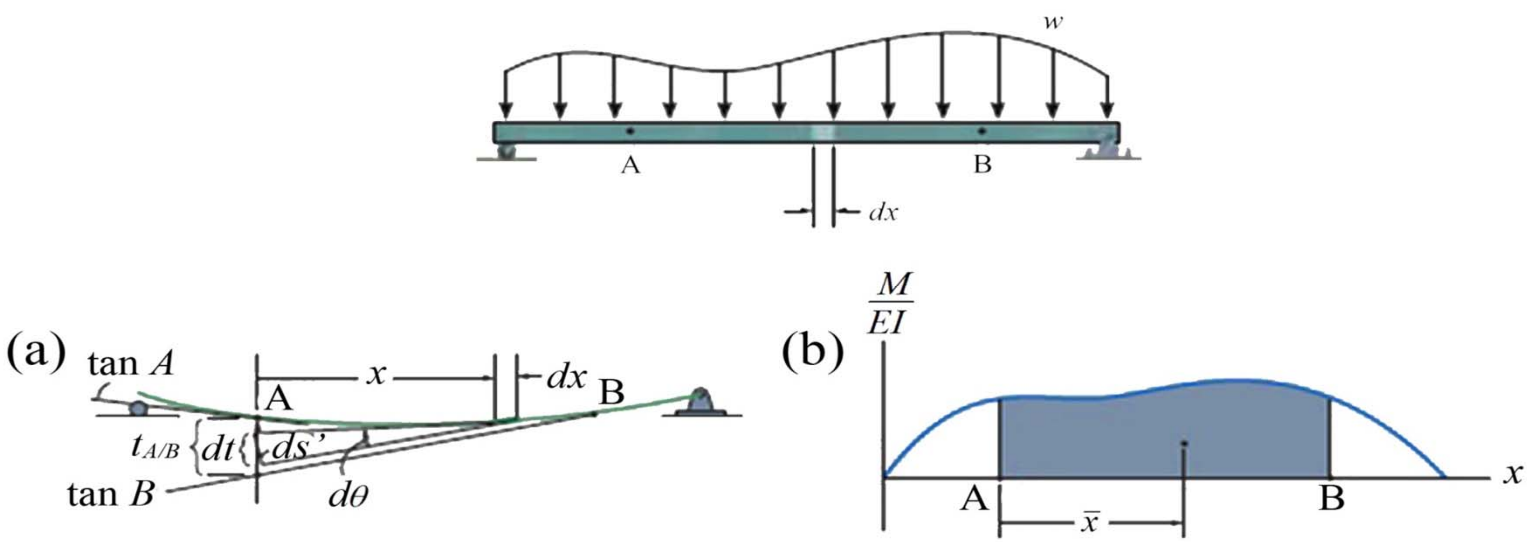
$$\theta_{B/A} = \int_A^B \frac{M}{EI} dx$$





Moment Area Method

Theorem 2:

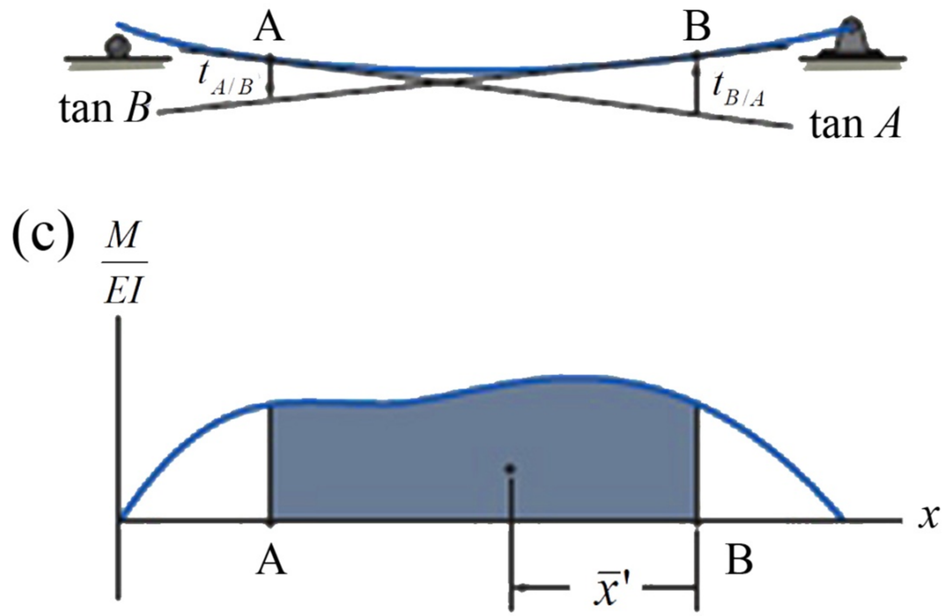


Moment Area Method

Theorem 2 (cont.):

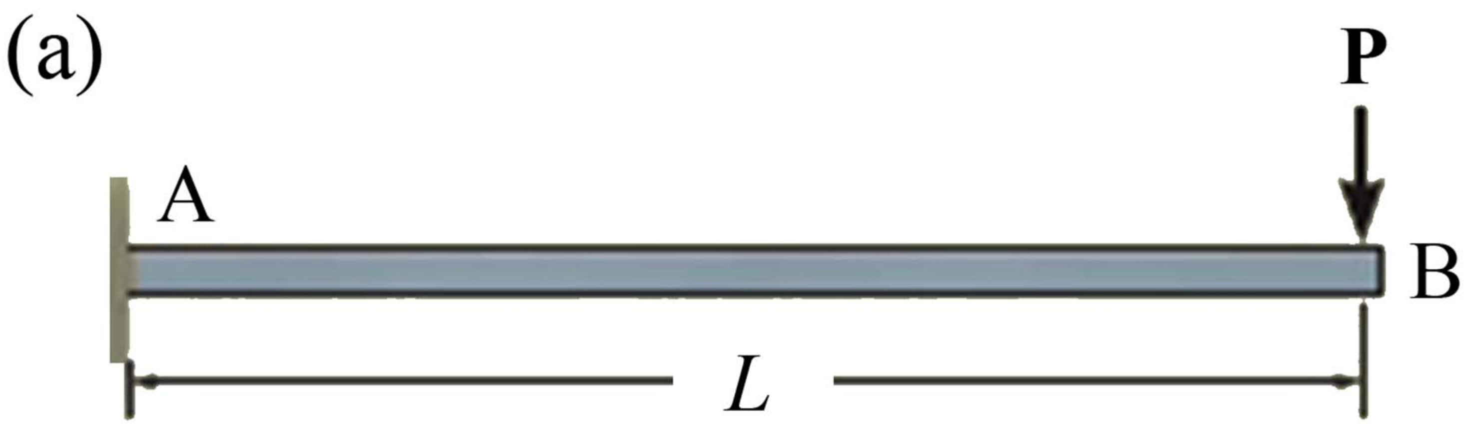
$$t_{A/B} = \int_A^B x \frac{M}{EI} dx$$

$$t_{A/B} = \bar{x} \int_A^B \frac{M}{EI} dx$$



Example 5

Determine the slope of the beam shown below at point *B*. *EI* is constant.

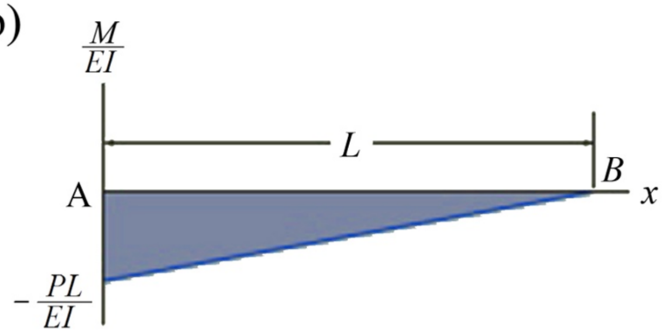


Example 5 (cont.)

Solutions

$$\theta_B = \theta_{B/A} \qquad \theta_C = \theta_{C/A}$$

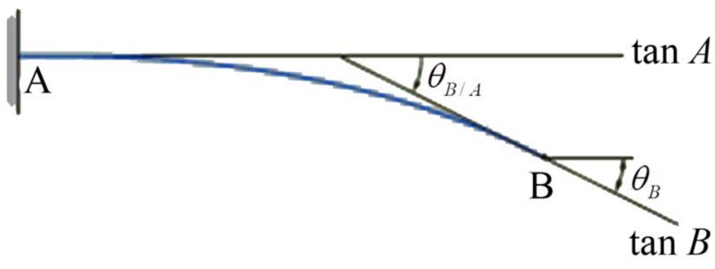
(b)



$$\theta_B = \theta_{B/A} = \left(-\frac{PL}{2EI}\right)\left(\frac{L}{2}\right) + \frac{1}{2}\left(-\frac{PL}{2EI}\right)\left(\frac{L}{2}\right) = -\frac{3PL^2}{8EI}$$

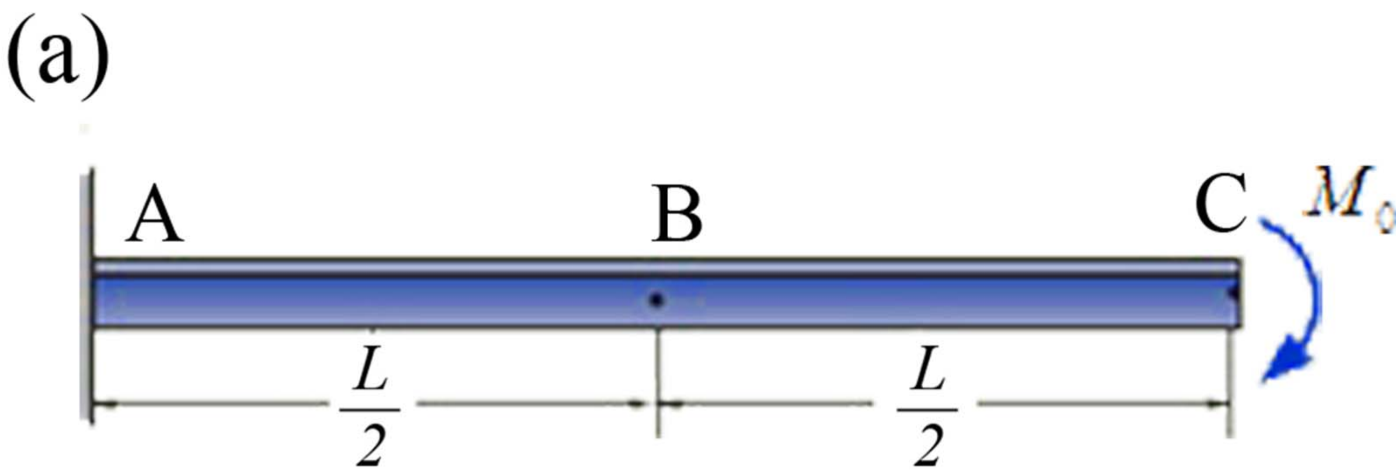
$$\theta_C = \theta_{C/A} = \frac{1}{2}\left(-\frac{PL}{2EI}\right)L = -\frac{PL^2}{2EI}$$

(c)



Example 6

Determine the displacement of points *B* and *C* of the beam shown below. *EI* is constant.



Example 6 (cont.)

Solutions

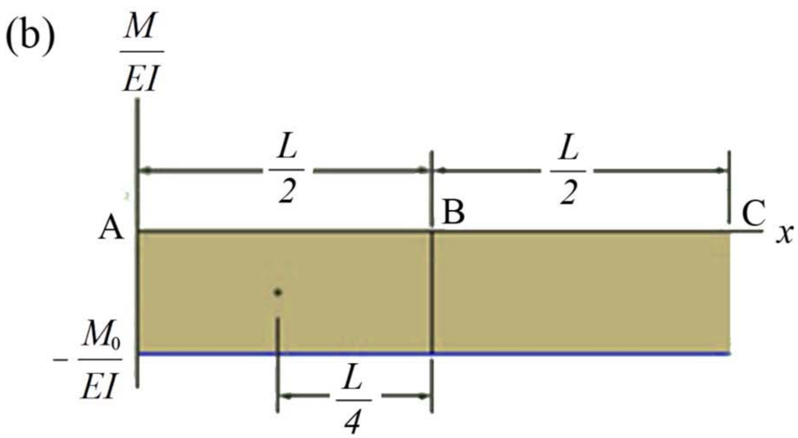
$$\Delta_B = t_{B/A} \qquad \Delta_C = t_{C/A}$$

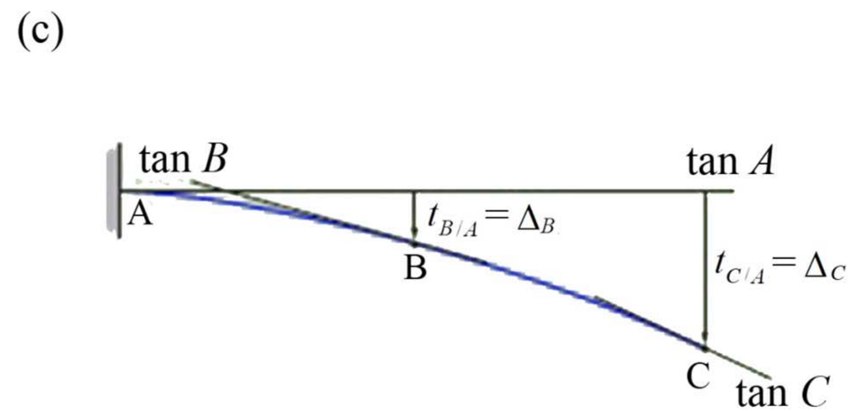
$$\Delta_B = t_{B/A} = \left(\frac{L}{4}\right)\left[\left(-\frac{M_0}{EI}\right)\left(\frac{L}{2}\right)\right] = -\frac{M_0 L^2}{8EI}$$

$$\Delta_C = t_{C/A} = \left(\frac{L}{2}\right)\left[\left(-\frac{M_0}{EI}\right)(L)\right] = -\frac{M_0 L^2}{2EI}$$

Example 6 (cont.)

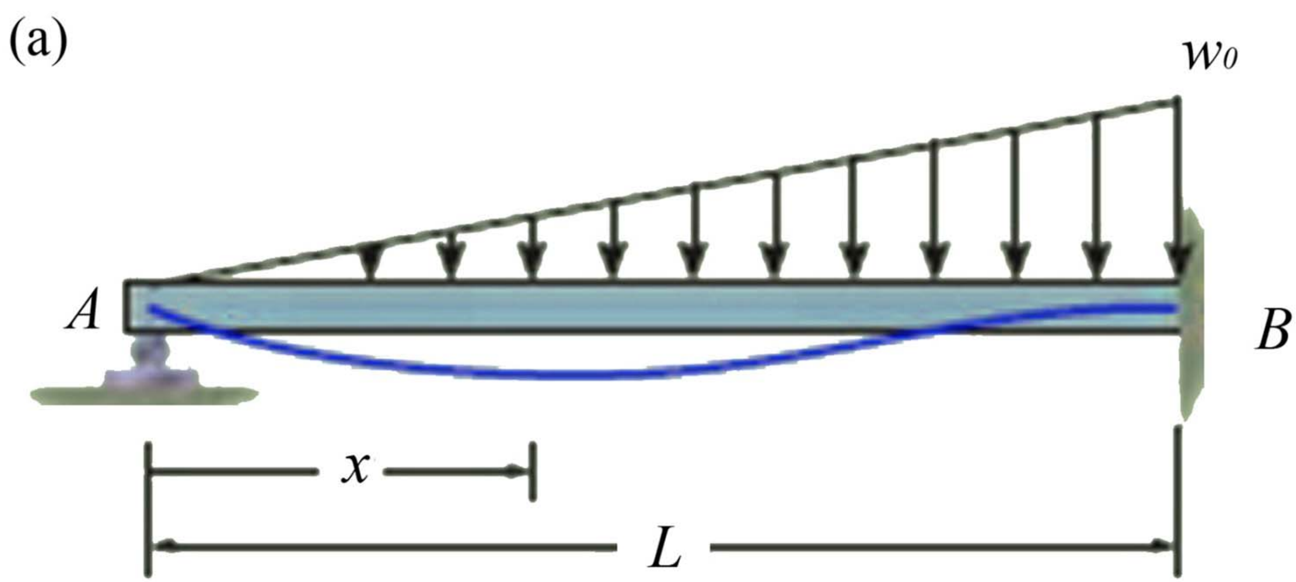
Solutions





Example 7

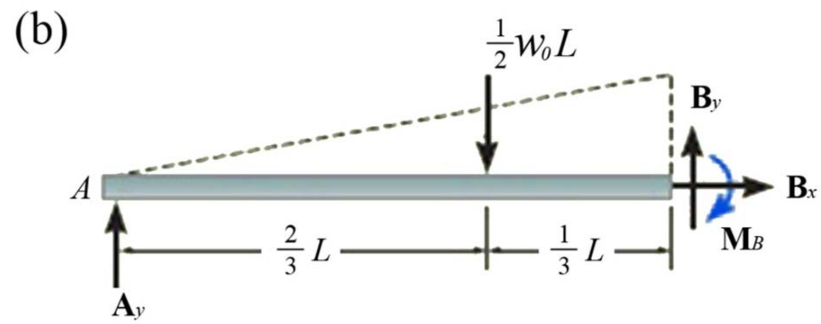
The beam is subjected to the distributed loading shown below. Determine the reaction at *A*. *EI* is constant.



Example 7 (cont.)

Solutions

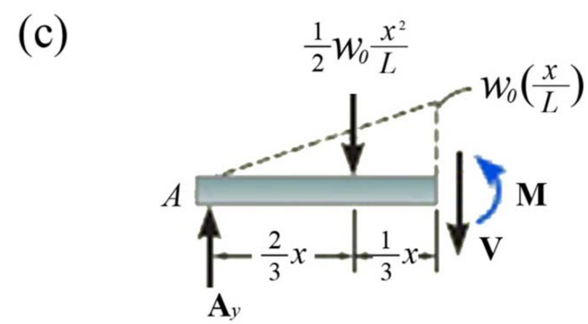
$$M = A_y x - \frac{1}{6} w_0 \frac{x^3}{L}$$



$$EI \frac{d^2 v}{dx^2} = A_y x - \frac{1}{6} w_o \frac{x^3}{L}$$

$$EI \frac{dv}{dx} = \frac{1}{2} A_y x^2 - \frac{1}{24} w_o \frac{x^4}{L} + C_1$$

$$EIv = \frac{1}{6} A_y x^3 - \frac{1}{120} w_o \frac{x^5}{L} + C_1 x + C_2$$



Example 7 (cont.)

Solutions

- boundary conditions: x = 0 and v = 0; x = L, dv/dx = 0; and x = L, v = 0.

$$x = 0,\ v = 0; \qquad 0 = 0 - 0 + 0 + C_2$$

$$x = L,\ \frac{dv}{dx} = 0; \qquad 0 = \frac{1}{2} A_y L^2 - \frac{1}{24} w_0 L^3 + C_1$$

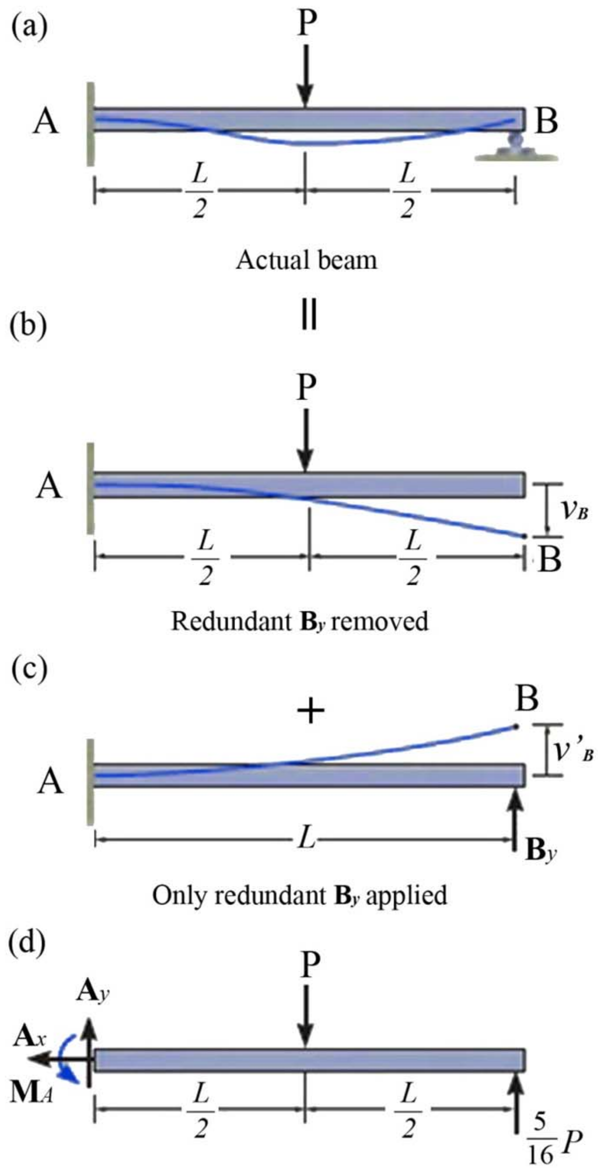
$$x = L,\ v = 0; \qquad 0 = \frac{1}{6} A_y L^3 - \frac{1}{120} w_0 L^4 + C_1 L + C_2$$

$$A_y = \frac{1}{10} w_0 L$$

$$C_1 = -\frac{1}{120} w_0 L^3 \qquad C_2 = 0$$

ocw.utm.my

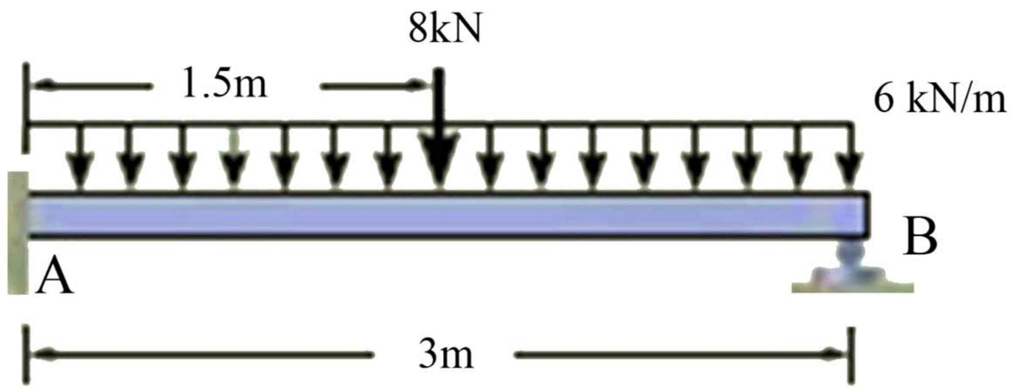
Use of the Method of Superposition



Example 8

Determine the reactions at the roller support *B* of the beam below, then draw the shear and moment diagrams. *EI* is constant.

(a)



Actual beam

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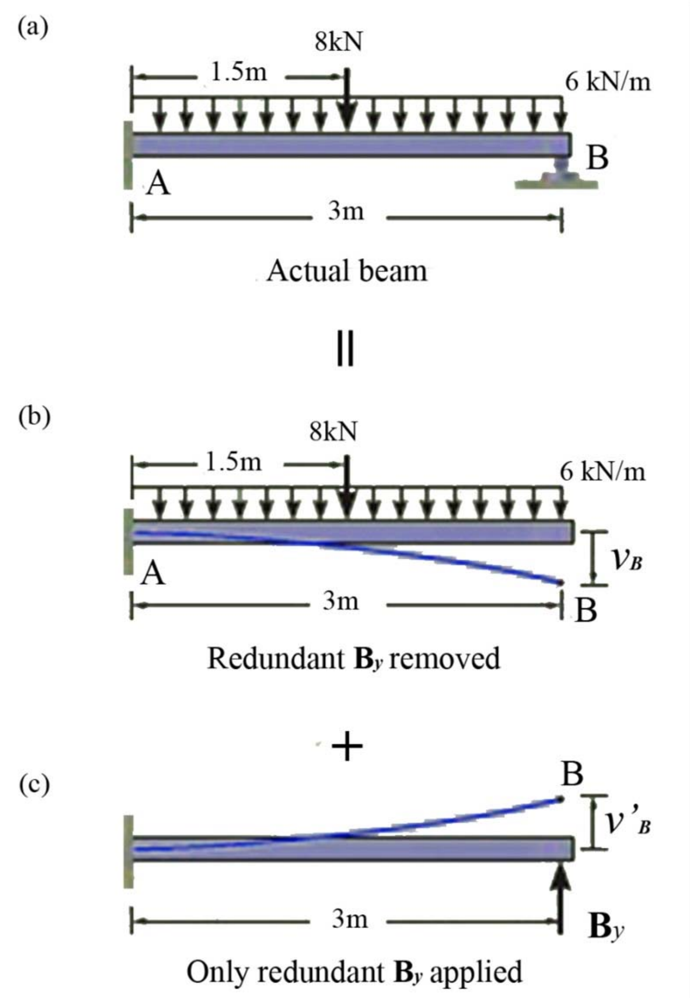
Example 8 (cont.)

Solutions

$$0 = v_B - v'_B \qquad (1)$$

$$v_B = \frac{wL^4}{8EI} + \frac{5PL^3}{48EI} = \frac{83.25\ \text{kN}\cdot\text{m}^3}{\text{EI}}$$

$$v'_B = \frac{PL^3}{3EI} = \frac{\left(9\ \text{m}^3\right)B_y}{EI}$$

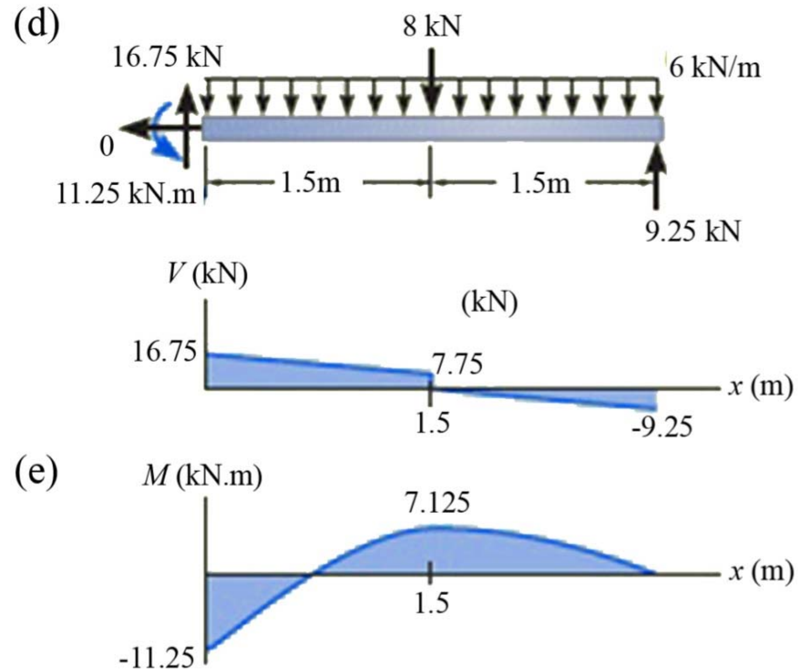


Example 8 (cont.)

Solutions

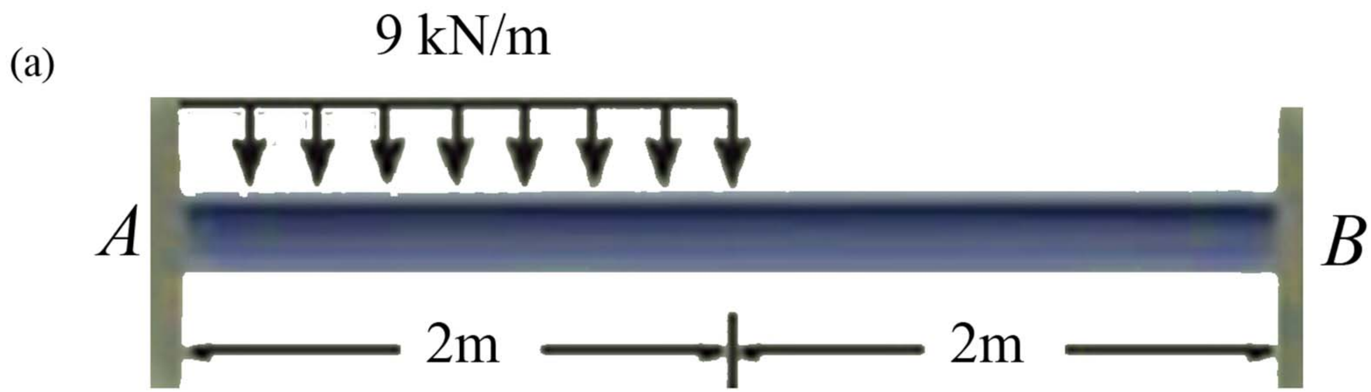
$$0 = \frac{83.25}{EI} - \frac{9B_y}{EI}$$

$$B_y = 9.25\,\text{kN}$$



Example 10

Determine the moment at *B* for the beam shown below. *EI* is constant. Neglect the effects of axial load.



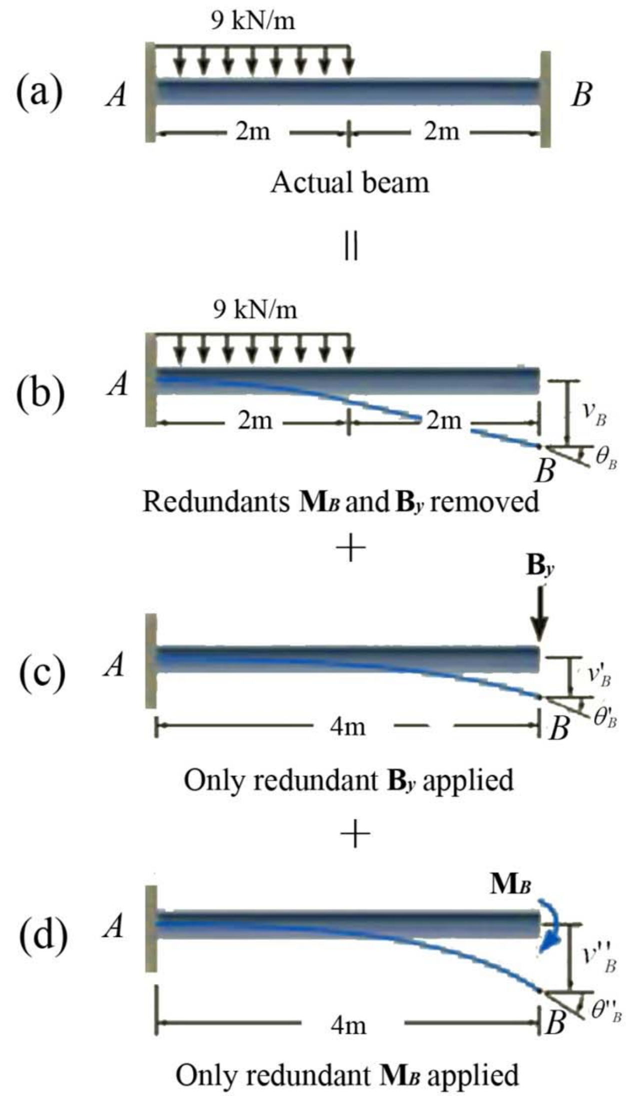
Actual beam

Example 10 (cont.)

Solutions

$$0 = \theta_B - \theta''_B \qquad (1)$$

$$0 = v_B + v'_B + v''_B \qquad (2)$$



Example 10 (cont.)

Solutions

$$\theta_B = \frac{wL^3}{48EI} = \frac{12\ \text{kN}\cdot\text{m}^3}{\text{EI}}$$

$$v_B = \frac{7wL^4}{384EI} = \frac{42\ \text{kN}\cdot\text{m}^3}{EI}$$

$$\theta'_B = \frac{PL^2}{2EI} = \frac{8B_y}{EI}$$

$$v'_B = \frac{PL^3}{3EI} = \frac{21.33B_y}{EI}$$

$$\theta''_B = \frac{ML}{EI} = \frac{4M_B}{EI}$$

$$v''_B = \frac{ML^2}{2EI} = \frac{8M_B}{EI}$$

Example 10 (cont.)

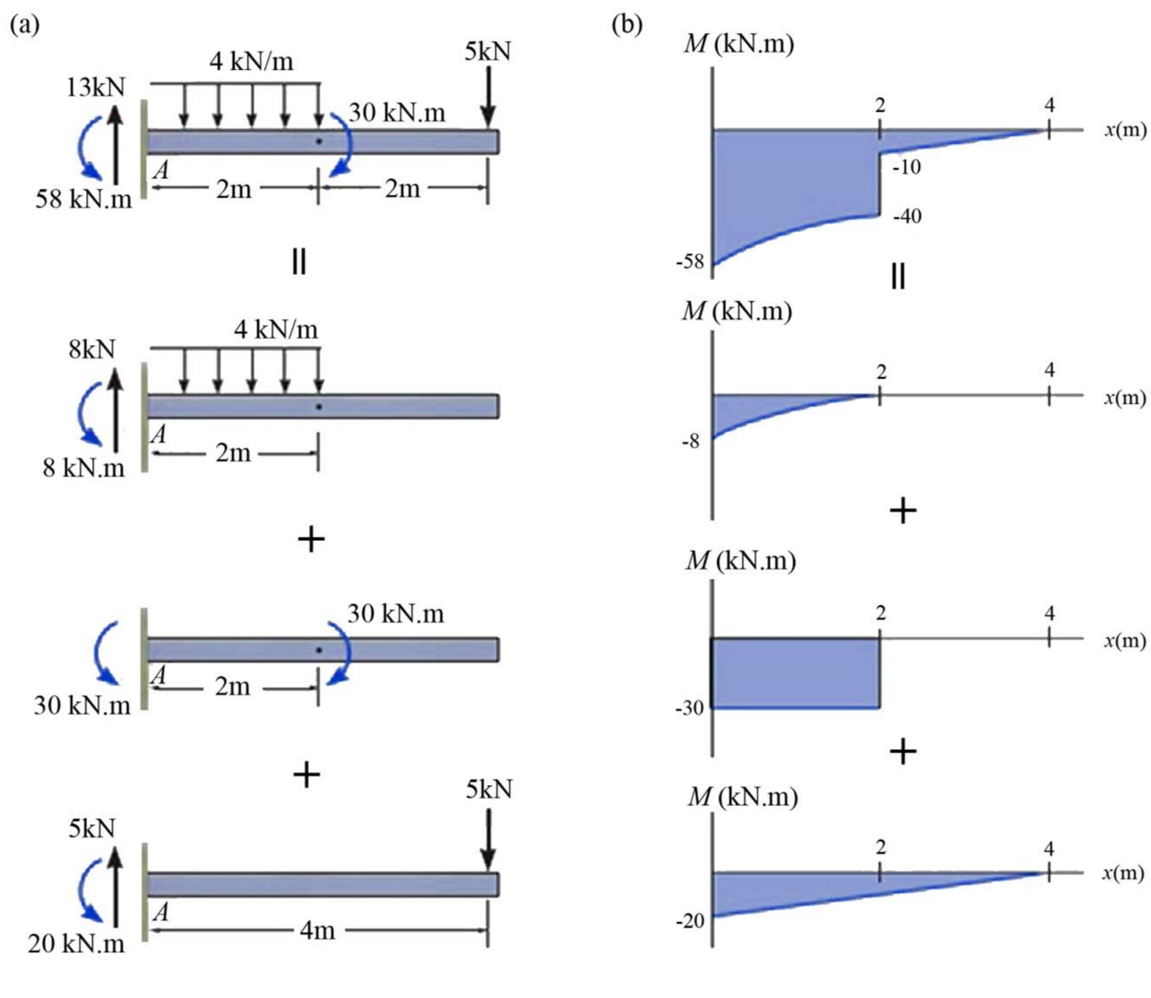
Solutions

$$0 = 12 + 8B_y + 4M_B$$
$$0 = 42 + 21.33B_y + 8M_B$$

$$B_y = 3.375 \text{ kN}$$
$$M_B = 3.75 \text{ kN} \cdot \text{m}$$

Use of the Moment-Area Method

Procedures:

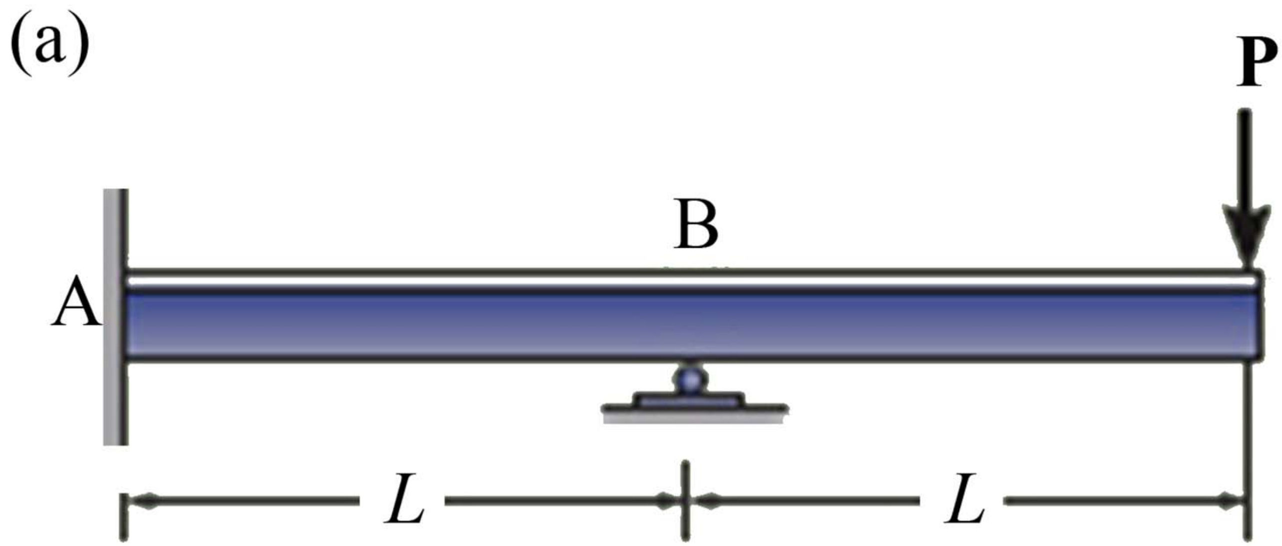


Superposition of loadings

Superposition of moment diagrams

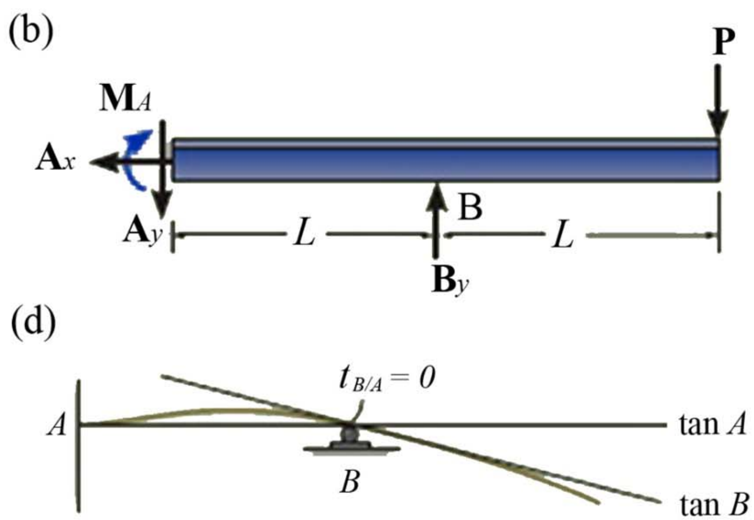
Example 11

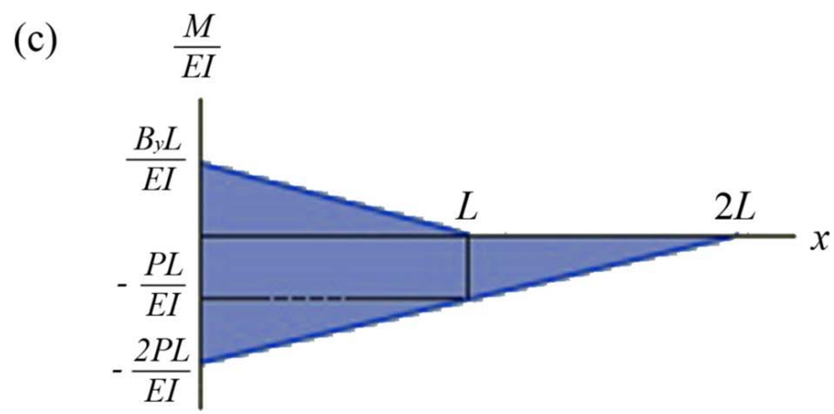
The beam is subjected to the concentrated force below. Determine the reactions at the supports. *EI* is constant.



Example 11 (cont.)

Solutions





Example 11 (cont.)

Solutions

$$t_{B/A} = \left(\frac{2}{3}L\right)\left[\frac{1}{2}\left(\frac{B_y L}{EI}\right)L\right] + \left(\frac{L}{2}\right)\left[\frac{-PL}{EI}(L)\right] + \left(\frac{2}{3}L\right)\left[\frac{1}{2}\left(-\frac{PL}{EI}\right)(L)\right] = 0$$

$$B_y = 2.5P$$

$$\sum F_x = 0; \qquad A_x = 0$$

$$\sum F_y = 0; \qquad -A_y + 2.5P - P = 0 \Rightarrow A_y = 1.5P$$

$$M_A = 0; \qquad -M_A + 2.5P(L) = P(2L) = 0 \Rightarrow M_A = 0.5PL$$

References

1. Hibbeler, R.C., Mechanics Of Materials, 8th Edition in SI units, Prentice Hall, 2011.
2. Gere dan Timoshenko, Mechanics of Materials, 3rd Edition, Chapman & Hall.
3. Yusof Ahmad, ‘Mekanik Bahan dan Struktur’ Penerbit UTM 2001