SKF 3143

Process Control and Dynamics:
Response of First Order Systems

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Learning Objectives

When I complete this chapter, I want to be able to do the following:

1. Analyze the response of the first order systems

Response of First-Order Systems

- The dynamic response of many processes and control system components can be represented by linear first-order differential equations.
- We refer to these processes as *first-order systems*.
- This section presents the response of first-order systems to three different types of input signals: a step function, a ramp, and a sine wave.
- The objective is to learn how the parameters of first-order systems affect their response so that later we can infer the important characteristics of the response of a system by simply examining its transfer function.
- First-order systems are also important because many higher-order systems can be treated as combinations of first-order systems in series and parallel.

Response of First-Order Systems

- Consider the linear first-order differential equation:

$$a_1 \frac{dy_{(t)}}{dt} + a_0 y_{(t)} = bx_{(t)} + c \qquad (6.1)$$

where *y* is the output or dependent variable; *x* is the input variables; *t* is time, the independent variable; and the parameters a_1, a_0, *b*, and *c* are constant.

- We can write the equation at the initial steady-state.

$$a_0 y_{(0)} = bx_{(0)} + c \qquad (6.2)$$

- Note that this equation establishes a relationship between the initial values of *x* and *y*. Subtracting Eq. 6.2 from Eq. 6.1 results in

$$a_1 \frac{dY_{(t)}}{dt} + a_0 Y_{(t)} = bX_{(t)} \qquad (6.3)$$

where

$$Y_{(t)} = y_{(t)} - y_{(0)}$$

$$X_{(t)} = x_{(t)} - x_{(0)}$$

are the deviation variables.

Response of First-Order Systems

- Eq. 6.3 is the general linear first-order differential equation in terms of the deviations of the input and output variables from their initial steady-state values.
- In process control it is customary to divide by the coefficient of the output variable, a_0.
- Such an operation results in the following equation called the *standard form* of the linear first-order differential equation.

$$\tau \frac{dY_{(t)}}{dt} + Y_{(t)} = KX_{(t)}$$

$$Y_{(s)} = \left[\frac{K}{\tau s + 1} \right] X_{(s)}$$

where

$\tau = \dfrac{a_1}{a_0}$ is the *time constant*

$K = \dfrac{b}{a_0}$ is the *steady-state gain*

Step Response

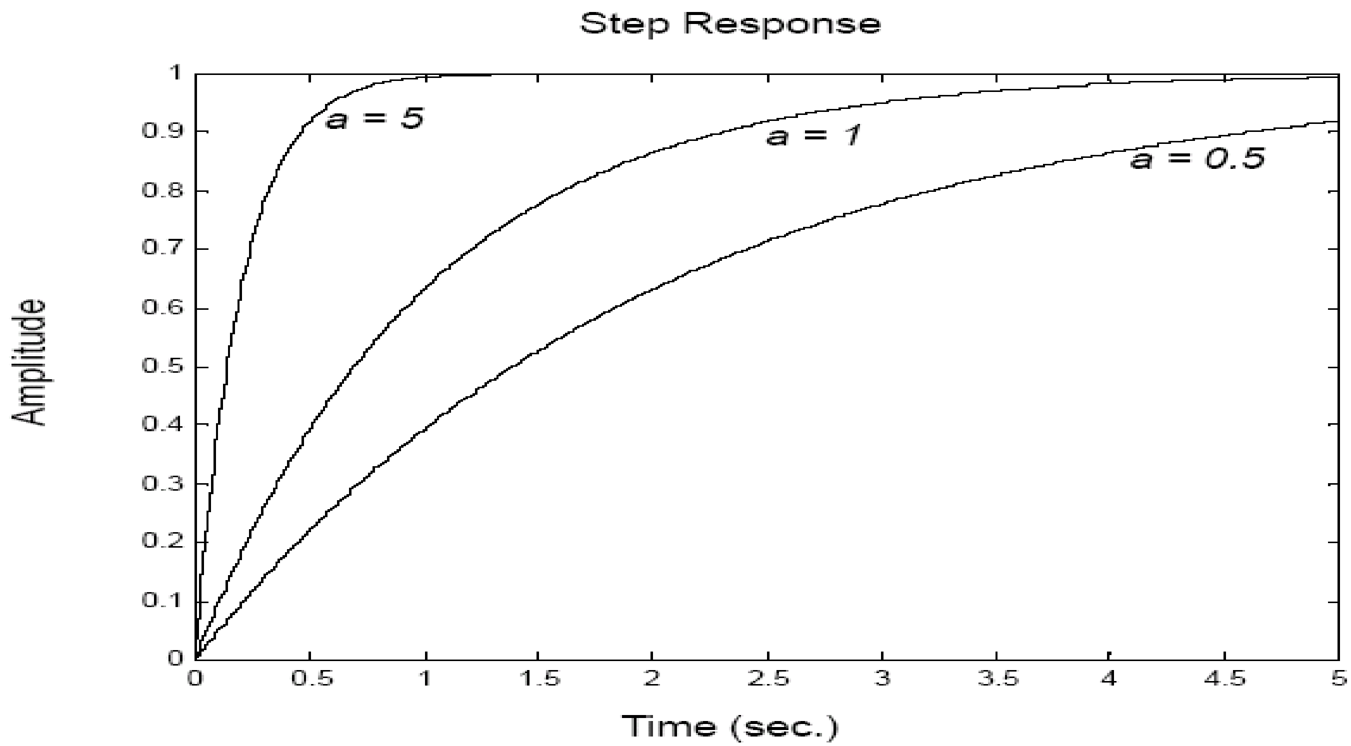
- To obtain the step response of magnitude *Δx*, we let $X_{(t)}=\Delta x$. The transform of the input is $X_{(s)}=\Delta x/s$. Substitute this to the equation of standard transfer function

$$Y_{(s)} = \frac{K}{\tau s+1}\frac{\Delta x}{s} = \frac{K\Delta x}{s(\tau s+1)}$$

- The time constant associated with this system is $\tau=1/a$. The time constant tells how quickly the system responds.
- For example, if $a=1$, then the system responds on the order of $\tau=1$ second; however, if $a=100$, then the system responds on the order of $\tau=0.01$ seconds.
- So, systems that respond quickly have *large* values of *a*, and systems that respond slowly have *small* values of *a*.
- The 2% settling time for a first order system is $T_s= 4\tau = 4/a$.
- This represents the amounts of time required for the system to reach and stay within 2% of the final value.

Step Response

- The plot below shows the step response for three different systems with *a* values of 0.5, 1, and 5. All systems attain a final value of 1, but note that the system with *a=5*, attains that value more *quickly* than the other two, and the system with *a=0.5* attains that value more *slowly* than the other two. The 2% settling times for the three system are 4/5, 4, and 8 seconds respectively.



Ramp Response

- A ramp is a linear increase in the input with time starting at time zero.
- The input function is given by $X_{(t)}=at$. The Laplace transform is $X_{(s)}=a/s^2$.
- Substitute this to the equation of standard transfer function and performing a partial fraction expansion yields

$$Y_{(s)} = \frac{K}{\tau s+1}\frac{a}{s^2} = \frac{\alpha_1}{\tau s+1} + \frac{\alpha_2}{s} + \frac{\alpha_3}{s^2}$$

- The Heaviside expansion gives

$$Y_{(s)} = \frac{Ka\tau^2}{\tau s+1} - \frac{Ka\tau}{s} + \frac{Ka}{s^2}$$

- Inverting yields

$$y_{(t)} = Ka\tau\left(e^{-t/\tau} - 1\right) + Kat$$

- The above expression has the interesting property that for large values of time ($t>>\tau$)

$$y_{(t)} \approx Ka(t-\tau)$$

Sinusoidal Response

- Processes are also subjected to periodic, or cyclic, disturbances. They can be approximated by a sinusoidal disturbance:

$$x_{\sin(t)} = \begin{cases} 0 & for \quad t < 0 \\ A\sin(\omega t) & for \quad t \geq 0 \end{cases}$$

where: A = Amplitude, ω = angular frequency

$$Y_{(s)} = \frac{KA\omega}{(\tau s+1)(s^2+\omega^2)}$$

$$= \frac{KA\omega}{(\tau s+1)(s^2+\omega^2)}\left(\frac{\omega\tau^2}{\tau s+1} - \frac{s\omega\tau}{s^2+\omega^2} + \frac{\omega}{s^2+\omega^2}\right)$$

Hence

$$y_{(t)} = \frac{KA}{\omega^2\tau^2+1}\left(\omega\tau e^{-t/\tau} - \omega\tau\cos\omega t + \sin\omega t\right)$$

or, by using trigonometric identities,

$$y_{(t)} = \frac{KA\omega\tau}{\omega^2\tau^2+1} e^{-t/\tau} + \frac{KA}{\sqrt{\omega^2\tau^2+1}}\sin(\omega t + \phi)$$

Response of Integrating Process Units

- Consider the model of liquid-level system with a pump attached to the outflow line.
- Assuming that the outflow rate *q* can be set at any time by the speed of the pump or by the valve in the effluent line:

$$A\frac{dh_{(t)}}{dt} = q_{i(t)} - q_{(t)}$$

- After subtracting the steady-state version of equation above,

$$A\frac{dh'_{(t)}}{dt} = q'_{i(t)} - q'_{(t)}$$

- Taking Laplace transforms

$$sAH'_{(s)} = Q'_{i(s)} - Q'_{(s)}$$

  and rearranging gives

$$H'_{(s)} = \frac{1}{As}\left[Q'_{i(s)} - Q'_{(s)}\right]$$

- Both of the transfer functions representing integrating units, characterized by the term *1/s*.

References:

- Seborg, D. E., Edgar, T. F., Mellinchamp, D. A. (2003). *Process Dynamics and Control*, 2nd. Edition. John Wiley, ISBN: 978-04-71000-77-8.
- Marlin, T. E. (2000). *Process Control: Designing Processes and Control System for Dynamic Performance*, 2nd. Edition. McGraw Hill, ISBN: 978-00-70393-62-2.
- Stephanopoulos, G. (1984). *Chemical Process Control. An Introduction to Theory and Practice*. Prentice Hall, ISBN: 978-01-31286-29-0