SEL4223 Digital Signal Processing

IIR Filter Design

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Digital Filter Design

- Filtering is a process of changing signal's spectral content. The change is usually to attenuate a range of frequencies in the signal while allowing the other frequencies to pass through.
- Below are how z-transform and DTFT are used to design the filter

Filter	Z-transform	DTFT
IIR (bilinear transformation)	• To convert an analog filter to the digital filter • To obtain difference equation	• To analyze the spectral response
FIR (windowing)	• *Not used. Difference equation can be obtained in time-domain by convolution*	• To analyze the spectral response

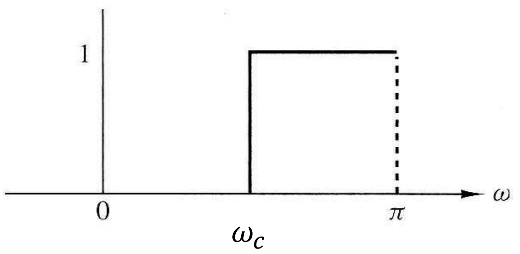
Digital Filter Design (cont.)

- Shown in the previous table are only for bilinear transformation and windowing techniques. There are many other techniques in designing both IIR and FIR filters. Different technique will use the z-transform and DTFT (or DFT) differently.

Ideal Filter

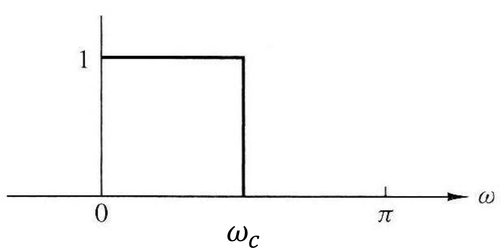
- Cut-off frequency (ω_c) is the only parameter considered.

Lowpass filter:



$$|H(\omega)|_{LP} = \begin{cases} 1 & for \quad \omega \leq \omega_c \\ 0 & for \quad \omega_c < \omega < \pi \end{cases}$$

Highpass filter:

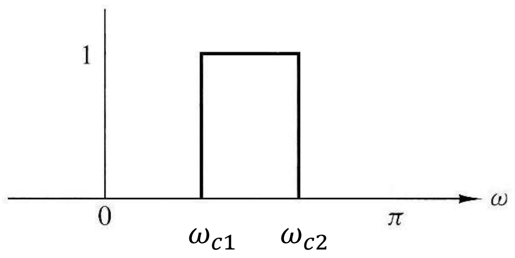


$$|H(\omega)|_{HP} = \begin{cases} 0 & for \quad \omega < \omega_c \\ 1 & for \quad \omega_c \leq \omega \leq \pi \end{cases}$$

$$= 1 - |H(\omega)|_{LP}$$

Ideal Filter (cont.)

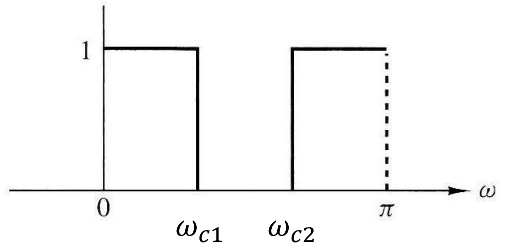
Bandpass filter:



$$|H(\omega)|_{HP} = \begin{cases} 0 & for \quad \omega < \omega_{c1} \\ 1 & for \quad \omega_{c1} \le \omega \le \omega_{c2} \\ 0 & for \quad \omega_{c2} < \omega \le \pi \end{cases}$$

$$= |H(\omega)|_{LP(\omega_{c2})} - |H(\omega)|_{LP(\omega_{c1})}$$

Bandstop filter:



$$|H(\omega)|_{HP} = \begin{cases} 1 & for \quad \omega \le \omega_{c1} \\ 0 & for \quad \omega_{c1} < \omega < \omega_{c2} \\ 1 & for \quad \omega_{c2} \le \omega \le \pi \end{cases}$$

$$= 1 - |H(\omega)|_{LP(\omega_{c2})} + |H(\omega)|_{LP(\omega_{c1})}$$

Non-ideal Filter

- Filter characteristic below must be considered:

ω_c - Cutoff frequency

ω_p - Passband edge frequency

ω_s - Stopband edge frequency

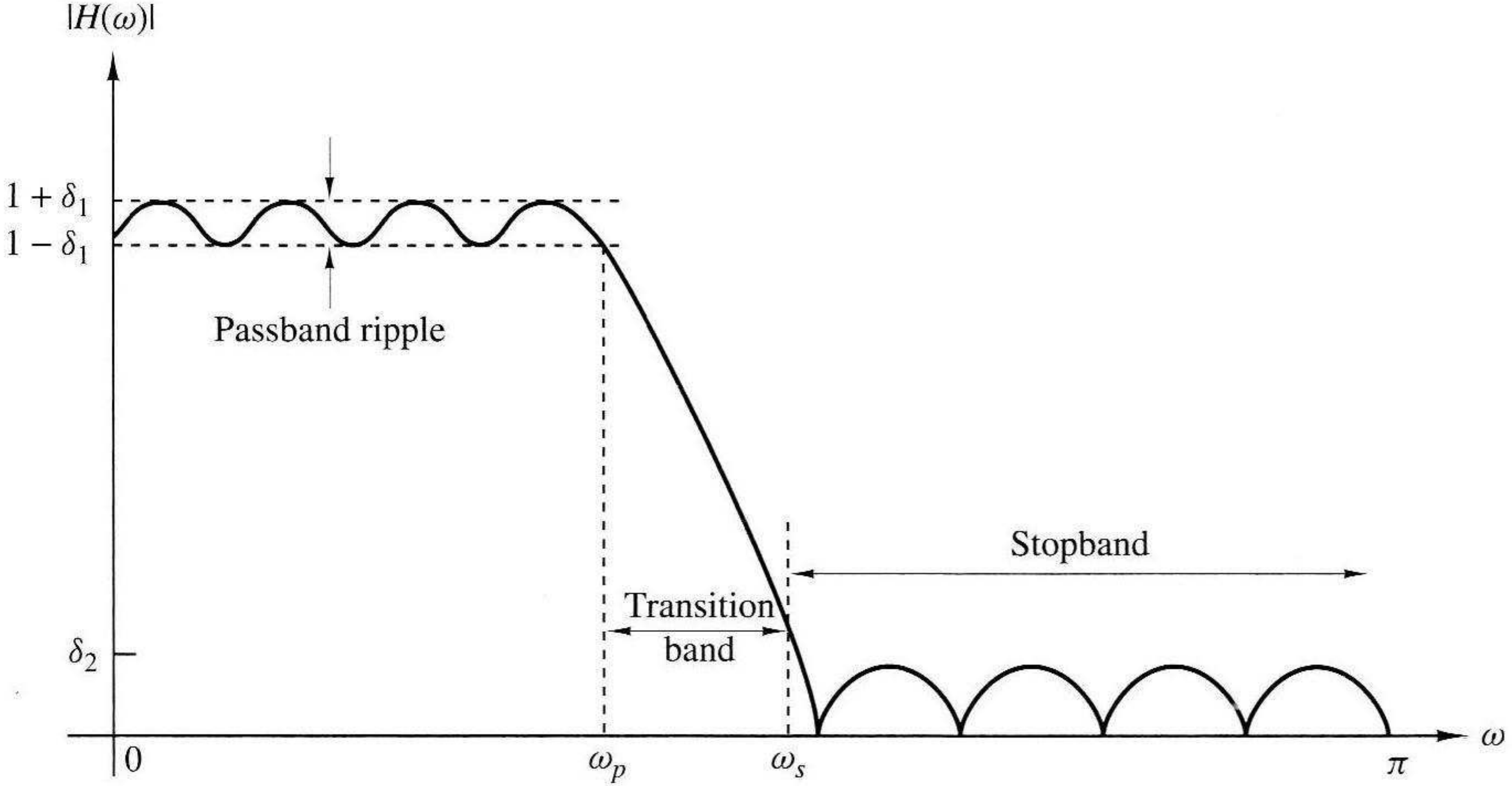
δ_p - Passband ripple

δ_s - Stopband ripple

N - Filter order

$\Delta\omega$ - Transition bandwidth

Non-ideal Filter (cont.)



IIR Filter Design

- There are two common technique used in designing the IIR filter
  - Impulse Invariance
  - Bilinear Transformation

- Basically, both techniques are implemented by converting system function of continuous-time filter ($H(s)$) to the discrete-time system function ($H(z)$). In other words, they map all poles in s-plane onto z-plane.

Bilinear Transformation

- In Bilinear transformation technique, relationship between the s-plane and z-plane is shown below where $c = \frac{2}{T_s}$ and T_s is the time sampling.

$$s = c\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

- Then, the relationship between the continuous-time frequency (Ω) and the discrete-time frequency (ω) is

$$\Omega = c.\tan\left(\frac{\omega}{2}\right) \quad \text{and} \quad \omega = 2tan^{-1}\left(\frac{\Omega}{c}\right)$$

Filter Design Procedure

1. Determine filter characteristic $(\delta_p, \delta_s, \omega_p, \omega_s, \omega_c, N)$:

   When designing filter, not all filter characteristics must be determine. Below are 3 ways of specifying the filter characteristics.

   I. Specify $\delta_p, \delta_s, \omega_p$ and ω_s

   II. Specify ω_c, N

   III. Specify ω_c, ω_s and δ_s or ω_c, ω_p and δ_p

2. Find system function of the continuous-time filter, $H(s)$:

   For Butterworth filter, need to find Ω_c and N.

3. Transform the continuous-time filter, $H(s)$ to the discrete-time filter, $H(z)$

4. Obtain the time-domain representation of the discrete-time filter for implementation:

   Either as an impulse response or as a difference equation.

Butterworth filter

- In this class, the filter design will be based only on Butterworth filter, which is one of the well known continuous-time filter. Another example of well known continuous-time filter is Chebyshev filter.
- The magnitude squared spectrum of continuous Butterworth filter is define as:

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

Butterworth filter (cont.)

- From there, it follows that

$$H(s)H(-s) = \frac{1}{1 + \left(\frac{-s^2}{{\Omega_c}^2}\right)^N}$$

- Based on the previous equation, it shows that Butterworth filter is an IIR filter as it contains poles at $s \neq 0$. From the equation, it also shows that Butterworth filter contains only poles and no zeros on the s-plane.

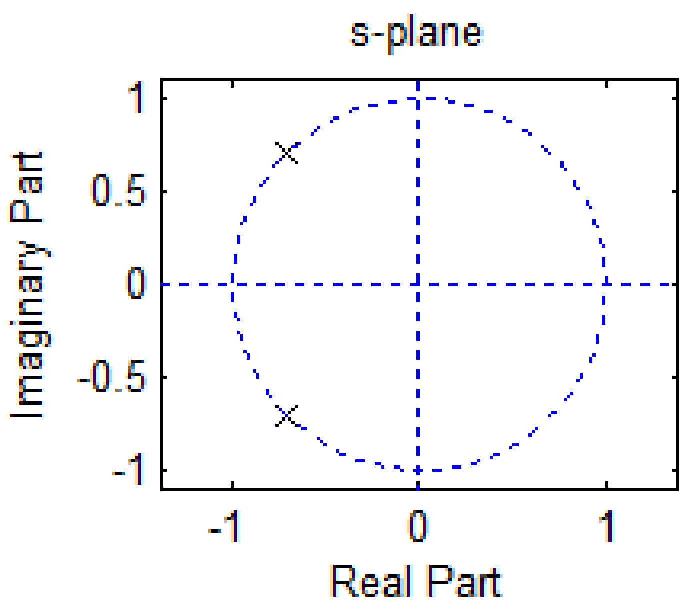
Butterworth filter (cont.)

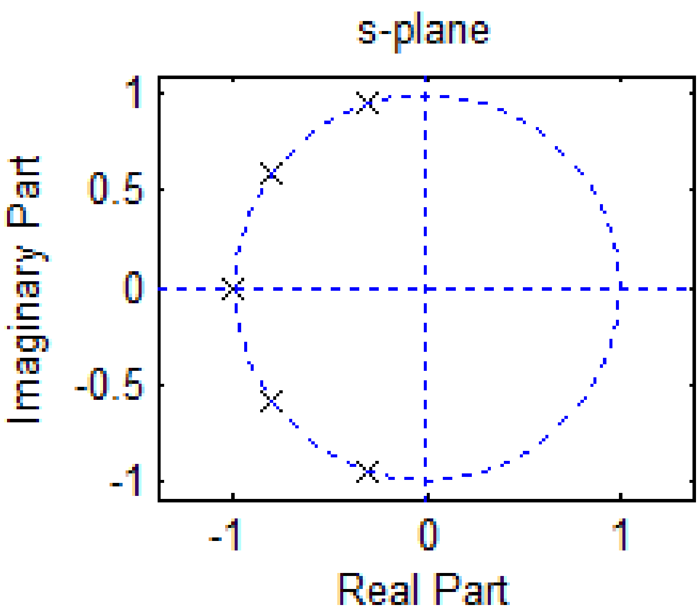
- The poles of the Butterworth filter can be determined as follow:

$$s_k = \Omega_c e^{j(2k+N+1)\pi/2N}, \qquad k = 0, 1, \dots, N-1$$

- Total number of the poles will be similar to *N* (filter order) where all poles are positions at $\sigma < 0$ on the s-plane. This is to ensure the causality and stability of the filter.
- The following figures are examples of the poles position on the s-plane with $\Omega_c = 1$.

Butterworth filter (cont.)





Butterworth filter (cont.)

- Then, the system function of the Butterworth filter is

$$H(s) = \frac{{\Omega_c}^N}{\prod_{k=0}^{N-1}(s - s_k)}$$

- To simplify the system function, always set $\Omega_c = 1$. Thus, the system function becomes

$$H(s) = \frac{1}{\prod_{k=0}^{N-1}(s - s_k)}$$

$$s_k = e^{j(2k+N+1)\pi/2N}, \qquad k = 0, 1, \dots, N-1$$

Butterworth filter (cont.)

- Below is table showing the system function for several filter order when $\Omega_c = 1$.

N	$H(s)$
1	$\frac{1}{s+1}$
2	$\frac{1}{s^2+1.4142s+1}$
3	$\frac{1}{(s+1)(s^2+s+1)}$
4	$\frac{1}{(s^2+0.7654s+1)(s^2+1.8478s+1)}$

Example 1

- Design a digital lowpass filter based on 2nd order Butterworth filter where cutoff frequency of the filter is $\omega_c = 0.5\pi\, rad$

Solution:

Step 1: Specify filter characteristics. Use given $\omega_c = 0.5\pi$ and $N = 2$.

Step 2: Find system function of the continuous filter by setting $\Omega_c = 1$, system function for 2nd order Butterworth filter is

$$H(s) = \frac{1}{s^2 + 1.4142s + 1}$$

Example 1 (cont.)

Step 3: Transform $H(s)$ to $H(z)$

$$s = c\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

Need to find c value. It can be computed based on given ω_c and $\Omega_\mathrm{c} = 1$ using equation below

$$\Omega_\mathrm{c} = c.\tan\left(\frac{\omega_c}{2}\right)$$

$$c = \frac{1}{\tan(0.25\pi)} = 1$$

Example 1 (cont.)

Then, the discrete-time system function is

$$H(z) = \frac{1}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 1.4142\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 1}$$

$$= \frac{(1+z^{-1})^2}{(1-z^{-1})^2 + 1.4142(1-z^{-1})(1+z^{-1}) + (1+z^{-1})^2}$$

$$= \frac{(1+z^{-1})^2}{3.4142 + 0.5858z^{-1}}$$

$$= 0.2929\frac{1+2z^{-1}+z^{-2}}{1+0.1864z^{-2}}$$

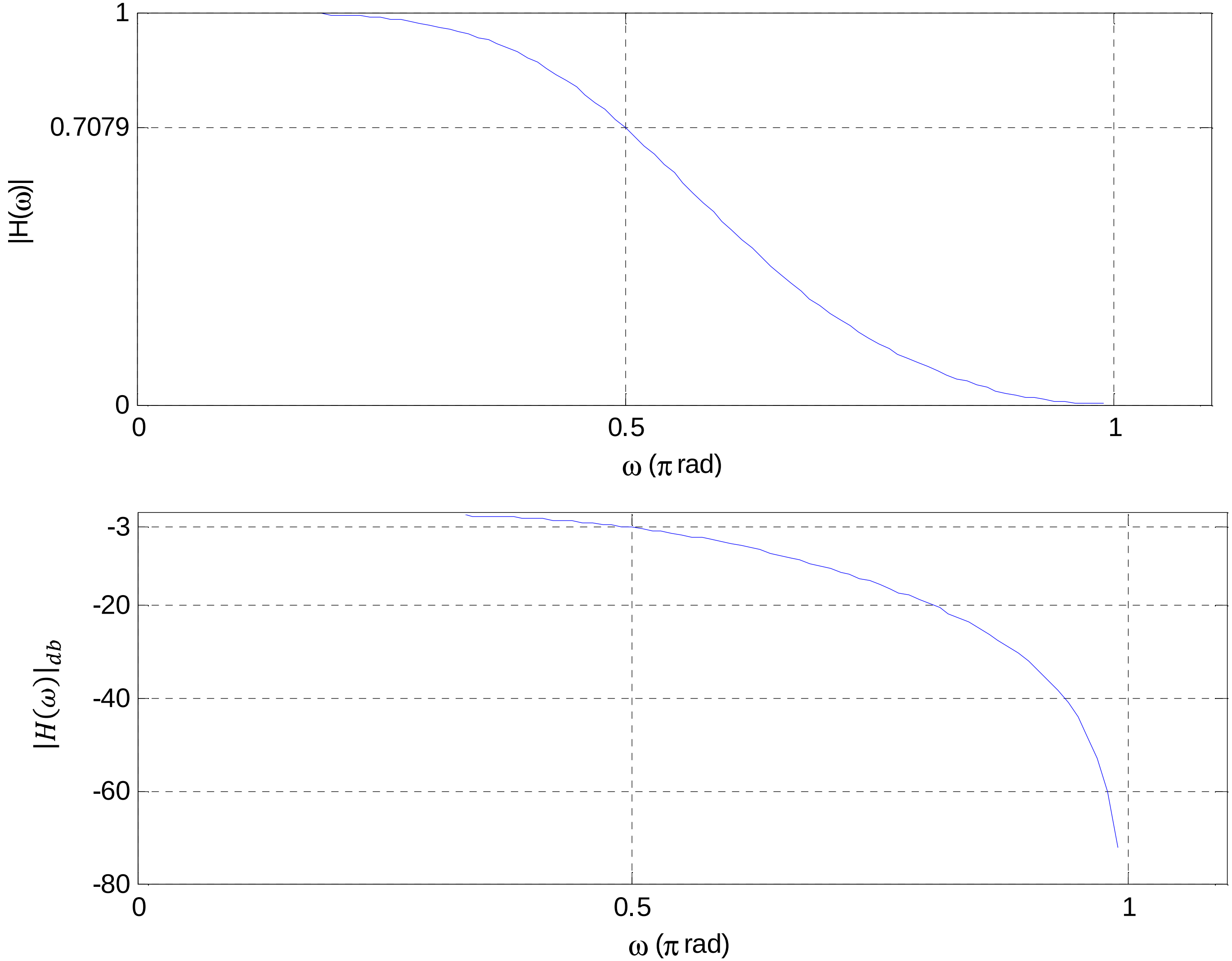
Example 1 (cont.)

Step 4: Obtain time-domain representation. Here we use difference equation.

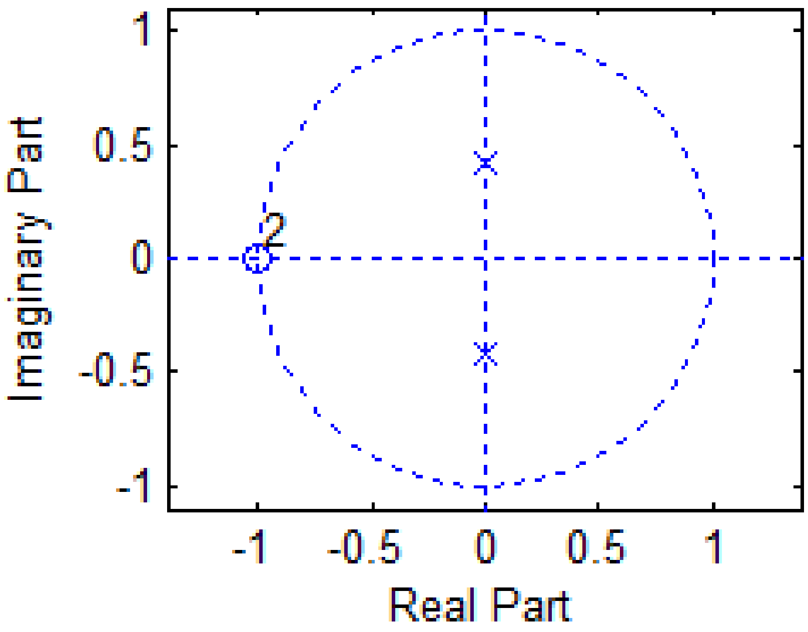
$$y[n] = 0.2929(x[n] + 2x[n-1] + x[n-2]) - 0.1864y[n-2]$$

- In order to see the shape of the filter, obtain and plot $|H(\omega)|$. For this example, the plot is shown below. Also shown is the magnitude dB plot and poles and zero plot for the filter.

Example 1 (cont.)



Example 1 (cont.)



Example 2

- Design an IIR lowpass filter based on Butterworth filter with the following filter characteristics.

$$\delta_s = \delta_p = 0.1$$

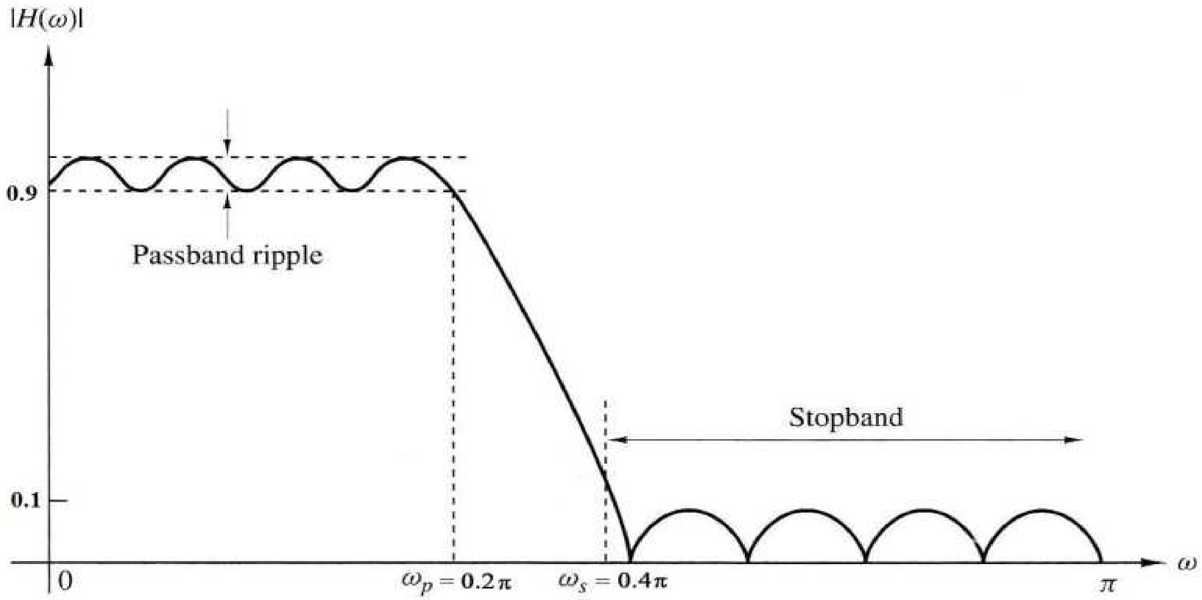
$$\omega_p = 0.2\pi$$

$$\omega_s = 0.4\pi$$

Solution:

Step 1: Specify filter characteristics. As given in the question the filter characteristics are shown in the following figure.

Example 2 (cont.)



Step 2: Find system function of the continuous filter

To obtain the system function, set $\Omega_c = 1$ and find N

Example 2 (cont.)

- From the filter characteristics, $|H(j\Omega)|^2$ at Ω_p and Ω_s can be identified, which are 0.9^2 and 0.1^2 respectively. Based on this information, *N* can be computed as follows where generally, the magnitude squared spectrum of Butterworth filter is set as $\Omega_\mathrm{c} = 1$

$$|H(j\Omega)|^2 = \frac{1}{1 + \Omega^{2N}}$$

- In order to compute N, evaluate the magnitude squared spectrum at Ω_p and Ω_s. Based on bilinear transformation;

$$\Omega = c.\tan\left(\frac{\omega}{2}\right)$$

Example 2 (cont.)

- $\Omega_\text{p} = c.\tan\left(\frac{\omega_p}{2}\right) = c.\tan\left(\frac{0.2\pi}{2}\right) = 0.3249c$
- $\Omega_\text{s} = c.\tan\left(\frac{\omega_s}{2}\right) = c.\tan\left(\frac{0.4\pi}{2}\right) = 0.7265c$

- Evaluating magnitude squared spectrum at Ω_p and Ω_s gets to

$$\frac{1}{1 + (0.3249\text{c})^{2N}} = 0.9^2 \qquad (1)$$

$$\frac{1}{1 + (0.7265\text{c})^{2N}} = 0.1^2 \qquad (2)$$

Example 2 (cont.)

- By manipulating and rearranging the two equations, it can be shown that

$$N = \frac{1}{2}\left(\frac{log\left(\frac{|H(\Omega_s)|^2.\left(1-|H(\Omega_p)|^2\right)}{|H(\Omega_p)|^2.\left(1-|H(\Omega_s)|^2\right)}\right)}{log\left(\frac{\Omega_p}{\Omega_s}\right)}\right)$$

$$= 3.7569$$

$$\approx 4$$

- Because N must be an integer number, value from the computation is round toward infinity to ensure the filter characteristics specified in step 1 is hold.

Example 2 (cont.)

Finally, with $\Omega_c = 1$ and $N = 4$, the system function of the continuous-time filter is

$$H(s) = \frac{1}{(s^2+0.7654s+1)(s^2+1.8478s+1)}$$

Step 3: Transform $H(s)$ to $H(z)$

$$s = c\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

Need to find c value. It can be computed based on equation (1) with $N = 4$. The results is $c = 2.5676$

Example 2 (cont.)

Based on the c value, the discrete-time system function is

$$H(z) = \frac{1}{\left(c^2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.7654c\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 1\right)\left(c^2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 1.8478c\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 1\right)}$$

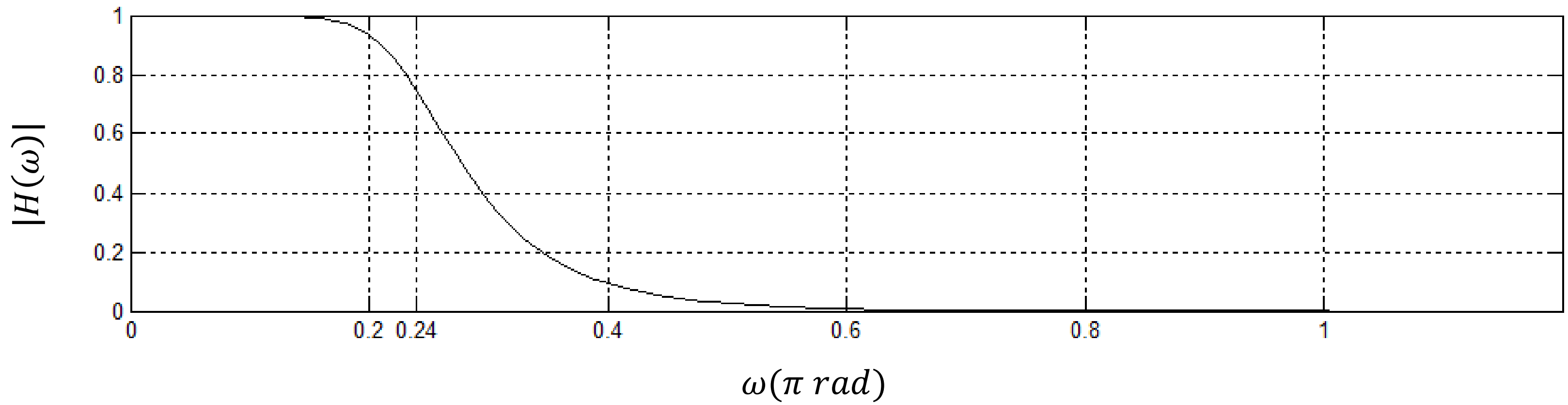
$$= \frac{\left(1+z^{-1}\right)^4}{(9.5578 - 11.1851z^{-1} + 5.6273z^{-2})(12.337 - 11.1851z^{-1} + 2.8481z^{-2})}$$

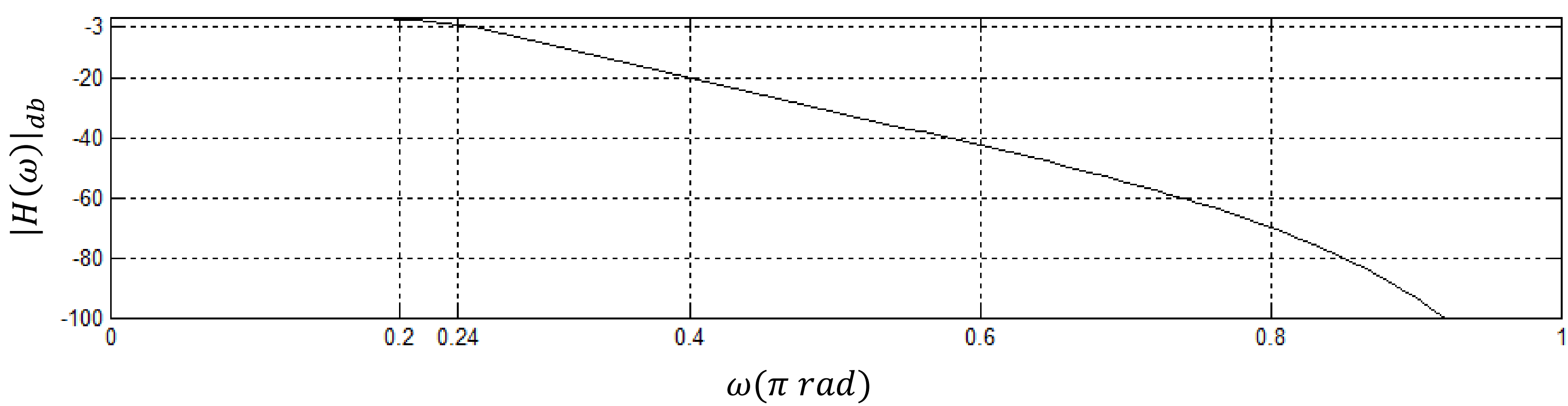
Step 4: Obtain time-domain representation. Do it yourself as an exercise.

- Magnitude spectrum, magnitude dB spectrum and pole-zero plot of the filter are shown below where the cutoff frequency is $\omega_c = 0.24$. ω_c can be computed using equation

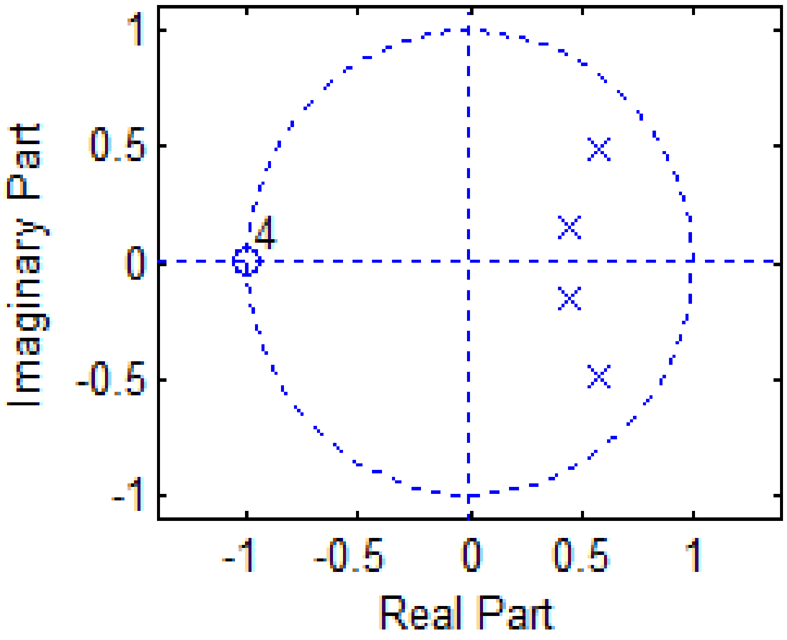
$$\omega = 2tan^{-1}\left(\frac{\Omega}{c}\right)$$

Example 2 (cont.)





Example 2 (cont.)



Pair of Poles Solution

- As in the Example 2, there are two pairs of poles ($N = 4$). Thus, denumerator of $H(z)$ is presented by multiplication of two sets of the 2nd order expressions.
- For each pair of poles, the transformation using the bilinear transformation from $H(s)$ to $H(z)$ where $\Omega_c = 1$ can be written as

$$H(s) = \frac{1}{s^2 + as + 1} \quad \Longrightarrow \quad H(z) = \frac{(1 + z^{-1})^2}{b_1 + b_2 z^{-1} + b_3 z^{-2}}$$

$$b_1 = c^2 + ac + 1$$
$$b_2 = -2c^2 + 2$$
$$b_3 = c^2 - ac + 1$$

Pair of Poles Solution (cont.)

- When N is odd, there will be one extra poles after pairing all conjugation poles. The transformation of the extra poles from $H(s)$ to $H(z)$ where $\Omega_c = 1$ can be written as

$$H(s) = \frac{1}{s+1} \quad \Longrightarrow \quad H(z) = \frac{(1+z^{-1})}{d_1 + d_2 z^{-1}}$$

$$d_1 = 1 + c$$

$$d_2 = 1 - c$$

Example 3

- Convert $H(s)$ to $H(z)$ for the 4th order Butterworth filter shown below using bilinear transformation. Assume $c = 1$

$$H(s) = \frac{1}{(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)}$$

Solution:

- From bilinear transformation, $H(z)$ can written as

$$H(z) = \frac{(1 + z^{-1})^4}{(b_1 + b_2 z^{-1} + b_3 z^{-2})(d_1 + d_2 z^{-1} + d_3 z^{-2})}$$

Example 3 (cont.)

where

$b_1 = 1^2 + 0.7654 + 1 = 2.7654$

$b_2 = -2 + 2 = 0$

$b_3 = 1^2 - 0.7654 + 1 = 1.2346$

$d_1 = 1^2 + 1.8478 + 1 = 3.8478$

$d_2 = -2 + 2 = 0$

$d_3 = 1^2 - 1.8478 + 1 = 0.1522$

Thus,

$$H(z) = \frac{(1 + z^{-1})^4}{(2.7654 + 1.2346z^{-2})(3.8478 + 0.1522z^{-2})}$$

Example 4

- Design an IIR Butterworth filter with $\omega_c = 0.5\pi$, $\omega_s = 0.9\pi$ and $\delta_s = 0.01$

Solution:

Step 1: Specify filter characteristics. Use filter characteristics as given in the question.

Step 2: Find system function of the continuous filter by computing N from the magnitude squared equation of the Butterworth filter where

$$\frac{1}{1 + {\Omega_s}^{2N}} = 0.01^2$$

Example 4 (cont.)

$$\Omega_s = c.\tan\left(\frac{\omega_s}{2}\right) = c.\tan\left(\frac{0.9\pi}{2}\right) = 6.3138c$$

- c value can be computed from equation below with $\Omega_c = 1$

$$\Omega_c = c.\tan\left(\frac{\omega_c}{2}\right)$$

- From there, $c = 1$. Thus $\Omega_s = 6.3138$ and the magnitude squared equation becomes

$$\frac{1}{1 + 6.3138^{2N}} = 0.01^2$$

Example 4 (cont.)

- Rearranging the magnitude squared equation leads to the formulation of N as below

$$N = \frac{1}{2} . \frac{\log\left(\frac{1}{\delta_s{}^2} - 1\right)}{log \Omega_s} = \frac{1}{2}\left(\frac{4}{0.8}\right) = 2.5 \approx 3$$

- Finally, with $\Omega_c = 1$ and $N = 3$, the system function of the continuous-time filter is

$$\mathrm{H(s)} = \frac{1}{(\mathrm{s}+1)(\mathrm{s}^2+\mathrm{s}+1)}$$

Example 4 (cont.)

Step 3: Transform $H(s)$ to $H(z)$

$$s = c\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

Thus,

$$\mathrm{H(z)} = \frac{(1+z^{-1})(1+z^{-1})^2}{(a_1+a_2z^{-1})(b_1+b_2z^{-1}+b_3z^{-2})}$$

where

$$a_1 = 1 + c = 2$$

$$a_2 = 1 - c = 0$$

Example 4 (cont.)

$$b_1 = c^2 + ac + 1 = 3.4142$$

$$b_2 = -2c^2 + 2 = 0$$

$$b_3 = c^2 - ac + 1 = 0.5858$$

Finally,

$$\mathrm{H(z)} = \frac{(1+z^{-1})^3}{2(3.4142 + 0.5858z^{-2})}$$

$$= 0.1464\frac{(1+z^{-1})^3}{(1+0.1716z^{-2})}$$

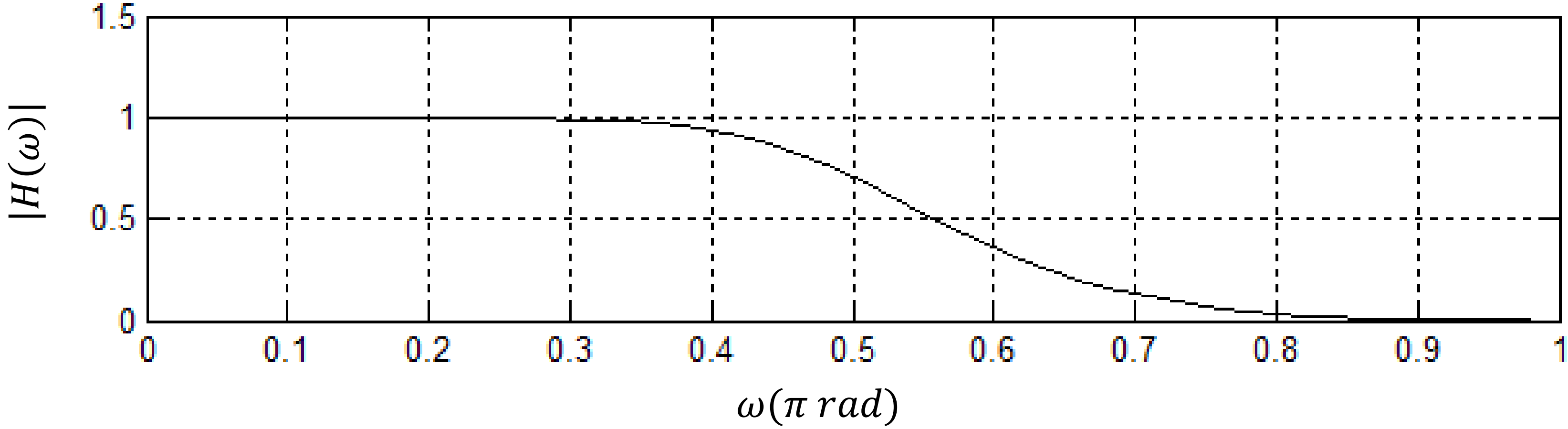
Example 4 (cont.)

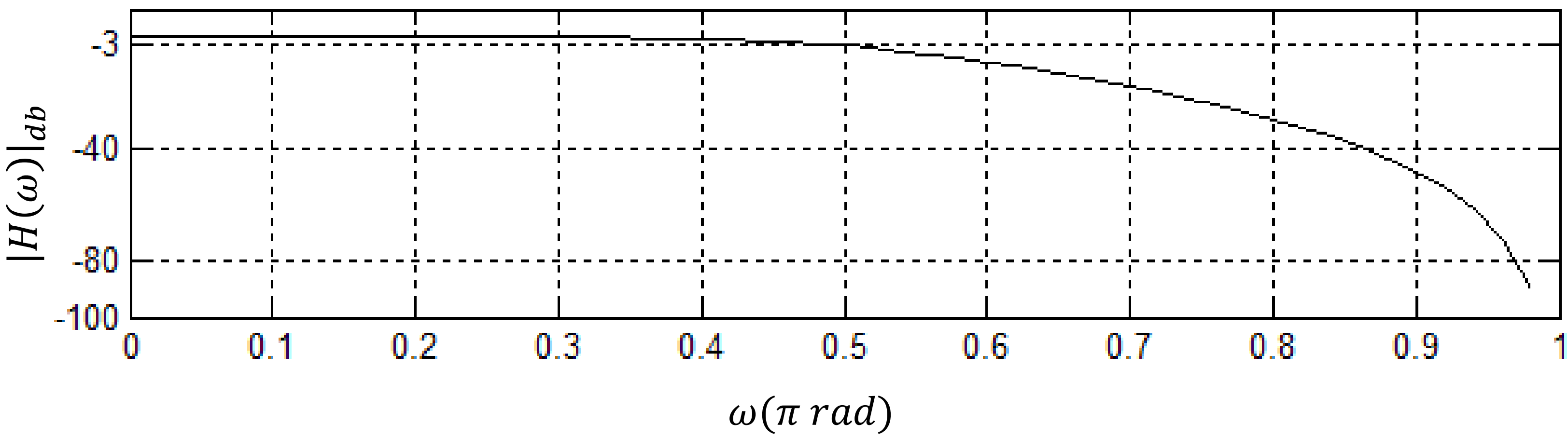
Step 4: Obtain time-domain representation.

$$H(z) = 0.1416\frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{1 + 0.1716z^{-2}}$$

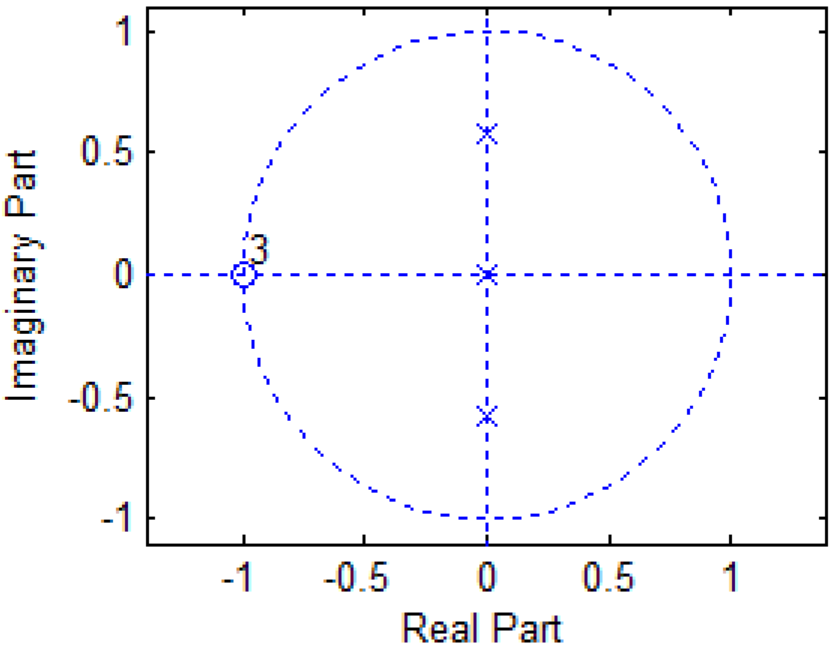
$$y[n] = 0.1416(x[n] + 3x[n-1] + 3x[n-2] + x[n-3]) \\ -0.1716y[n-2]$$

Example 4 (cont.)



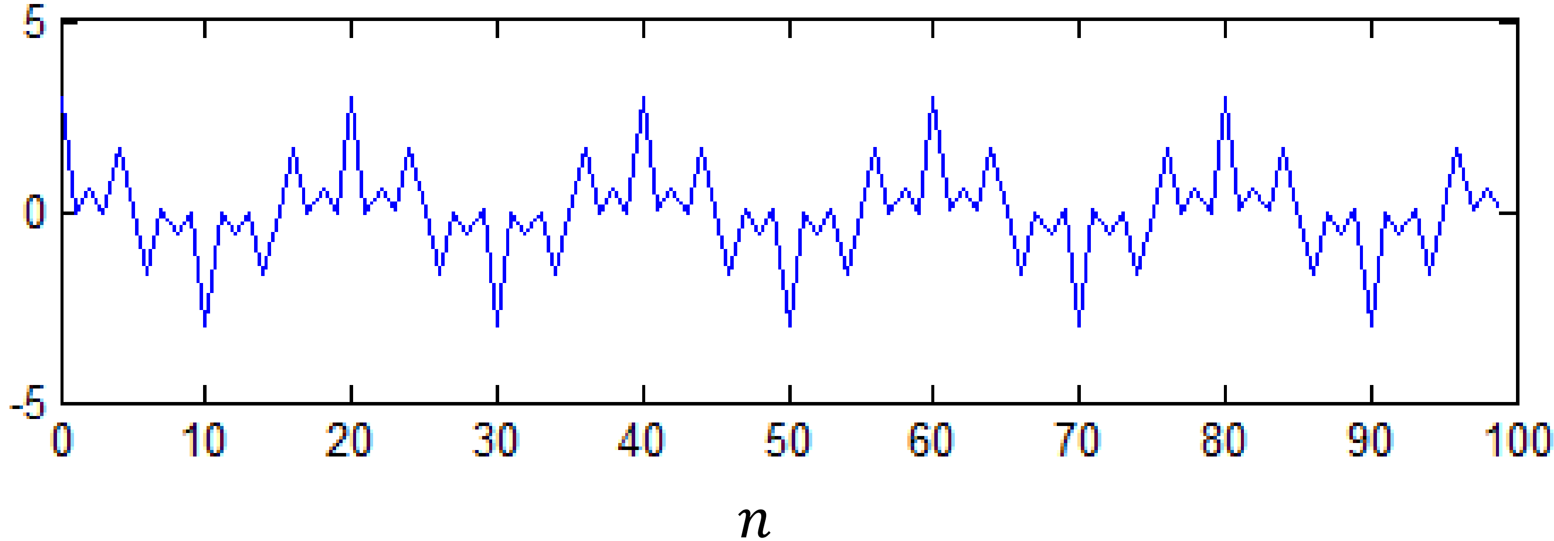


Example 4 (cont.)

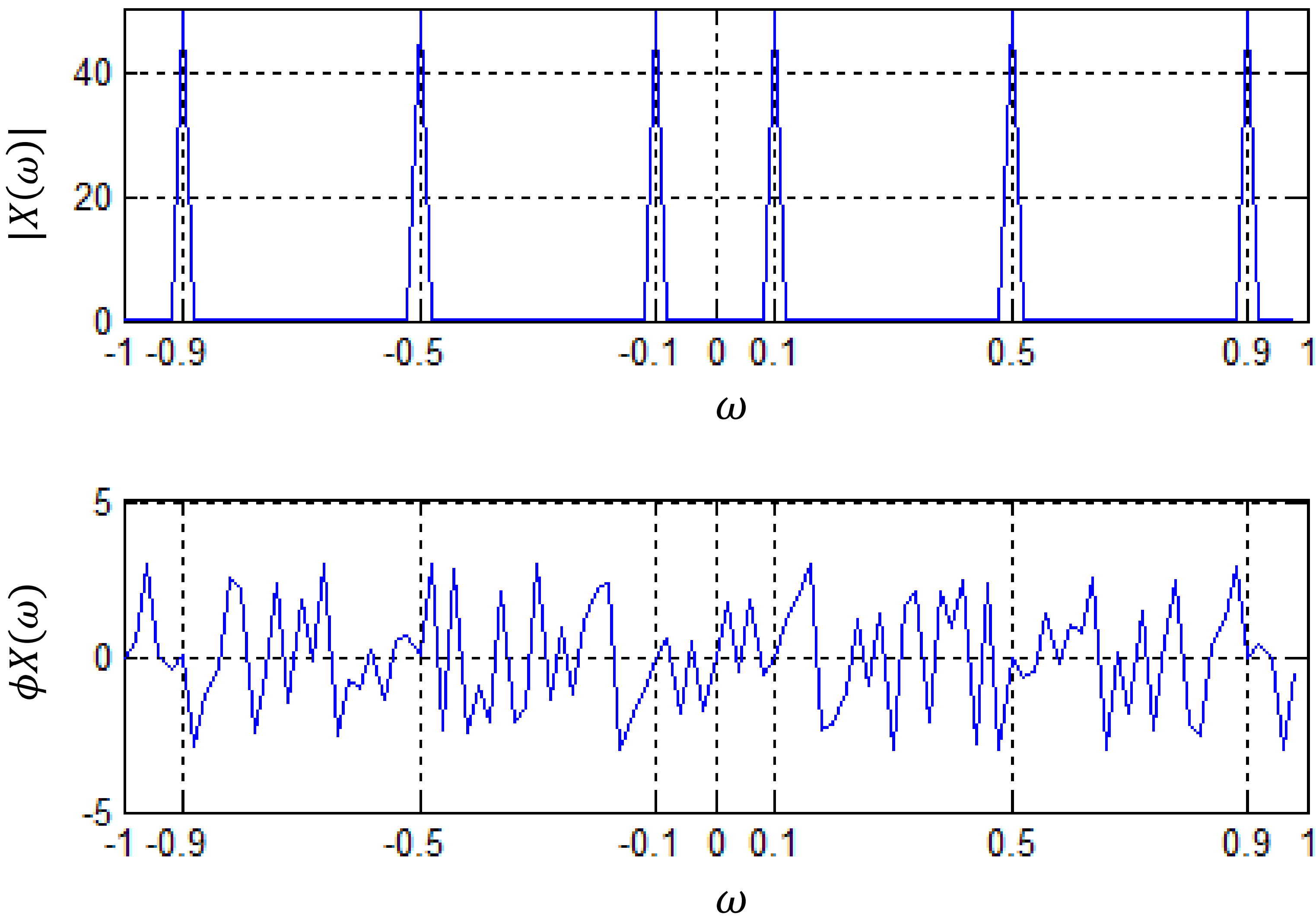


Example 5

- Design an IIR Butterworth filter that will attenuate frequencies component at $\omega = 0.5\pi$ and $\omega = 0.9\pi$ in signal $x[n]$ shown below. Also shown are the signal's magnitude and phase spectrum.



Example 5 (cont.)

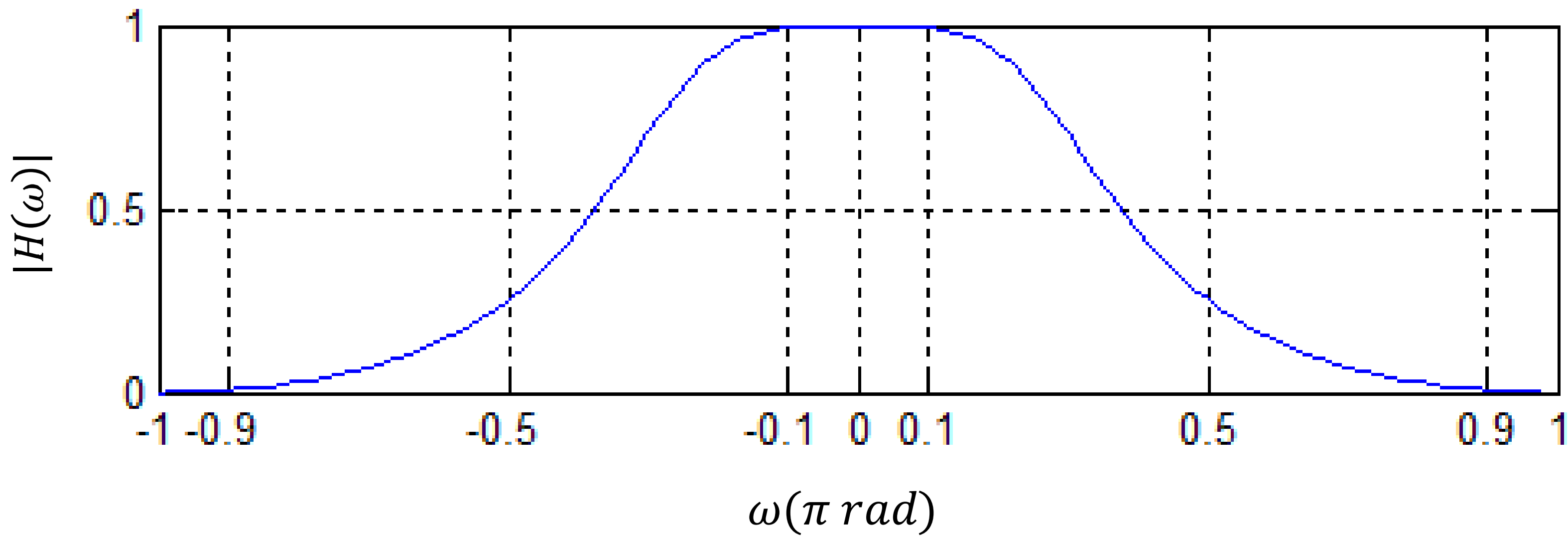


Example 5: 2^{nd} order BF

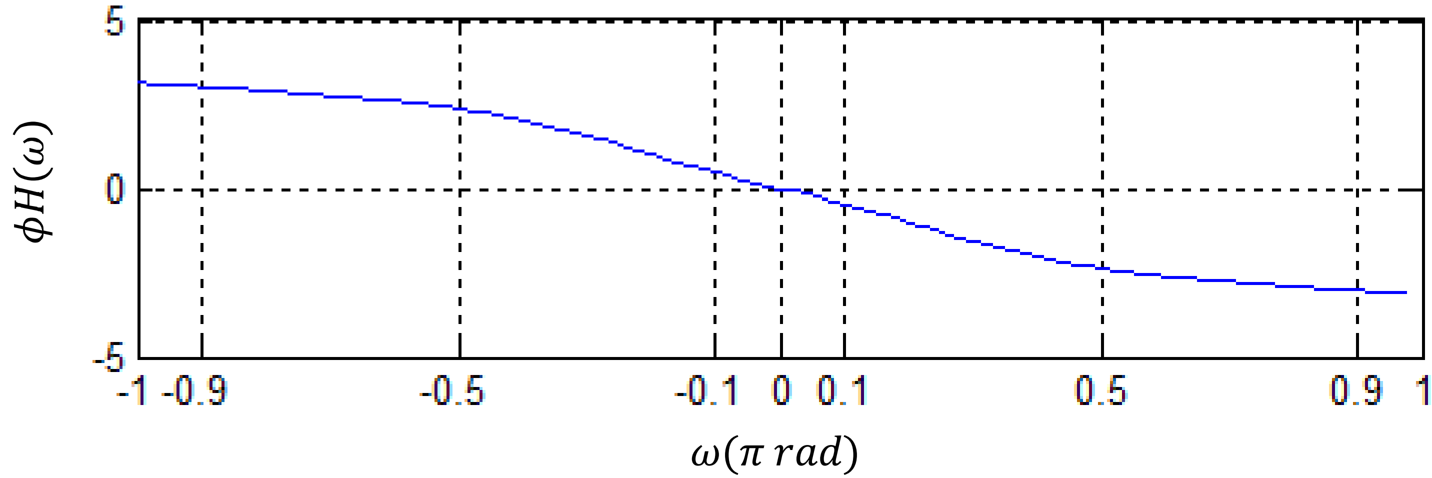
Solution:

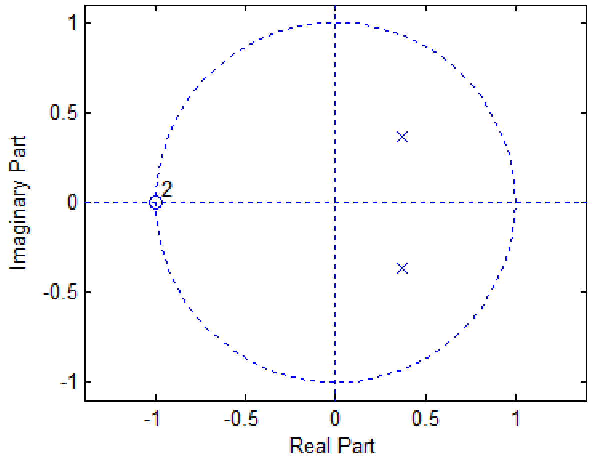
- Below are the solution by applying 2^{nd} order and 9^{th} order IIR Butterworth filter to signal $x[n]$ with $\omega_c = 0.3\pi$

- *2^{nd} order Butterworth filter:*



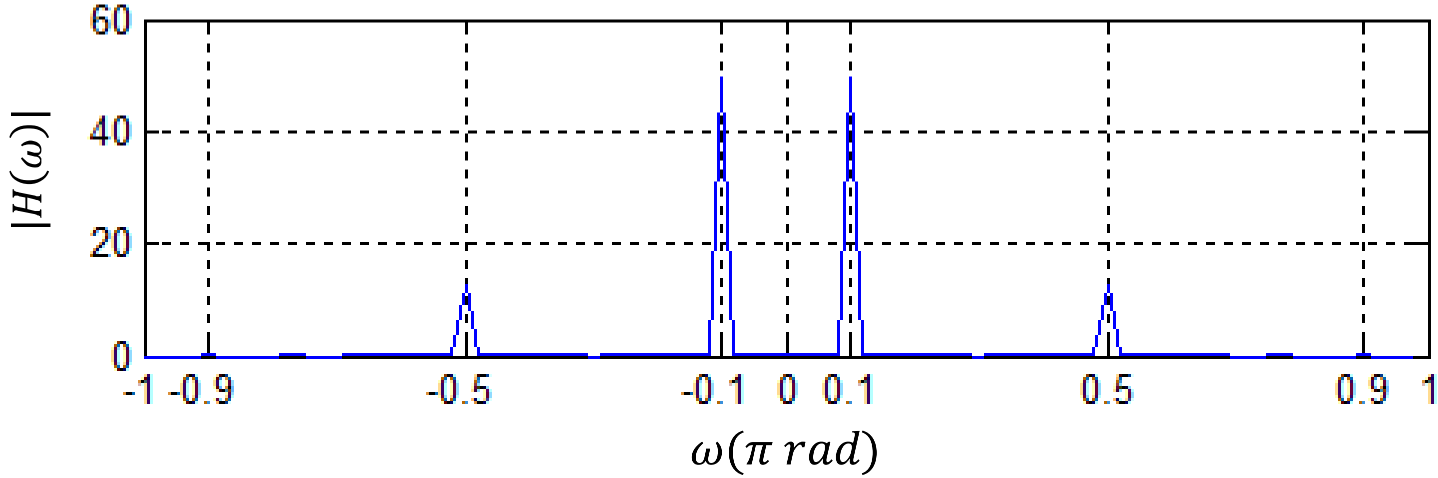
Example 5: 2nd order BF

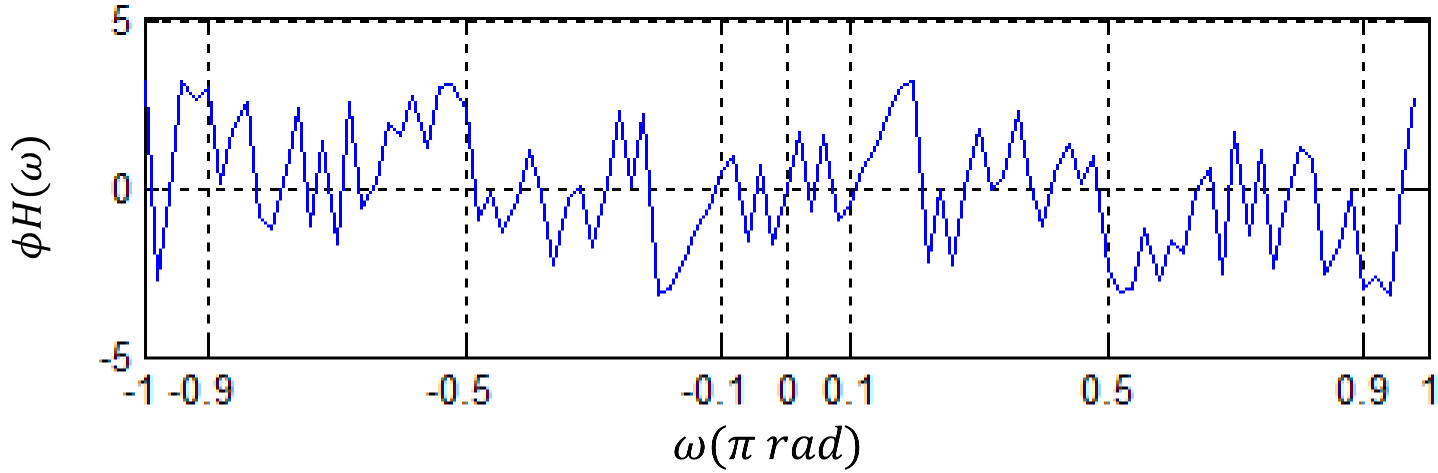




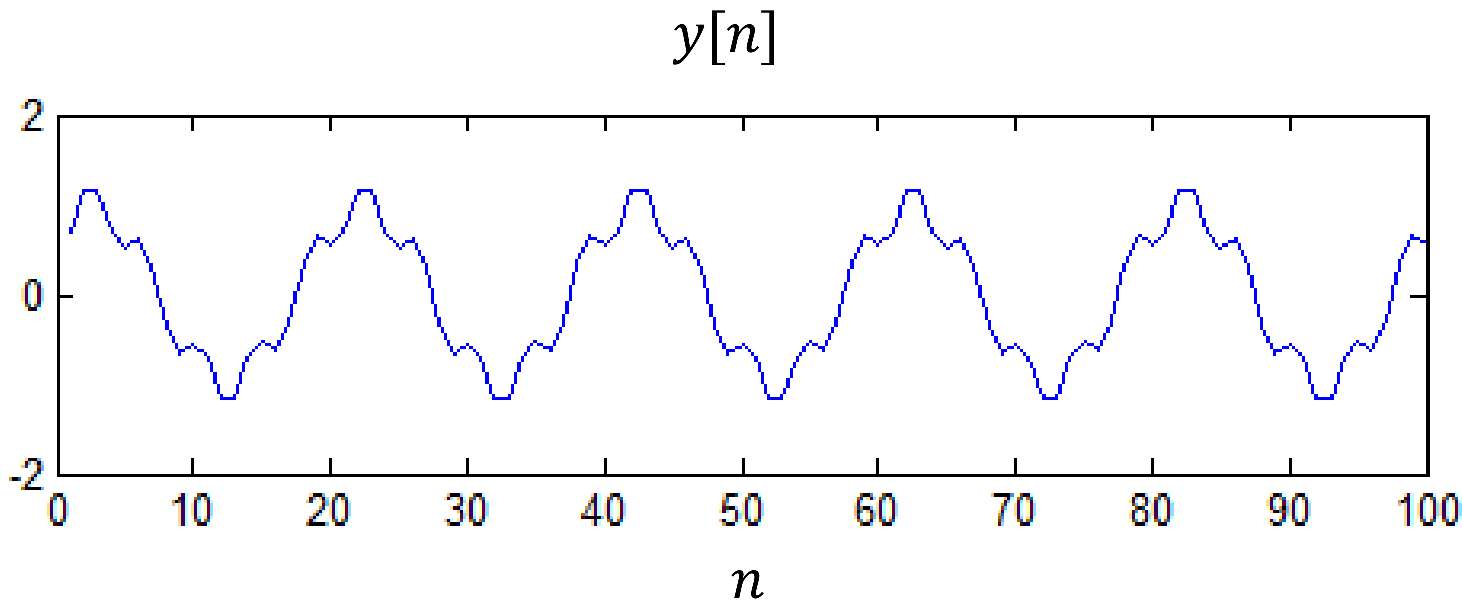
Example 5: 2nd order BF

- Output



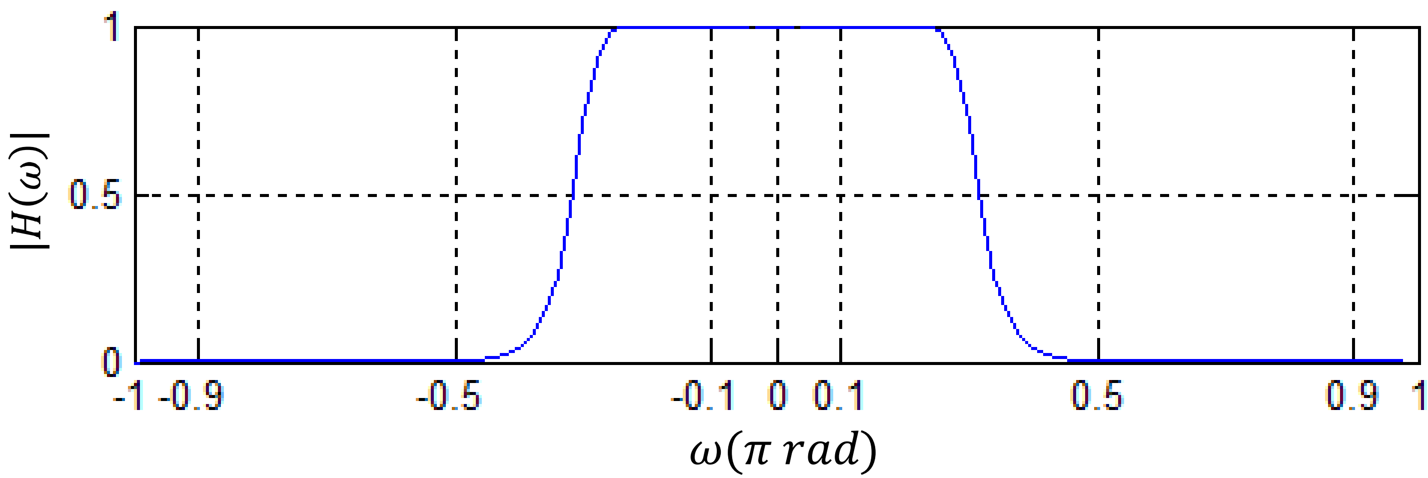


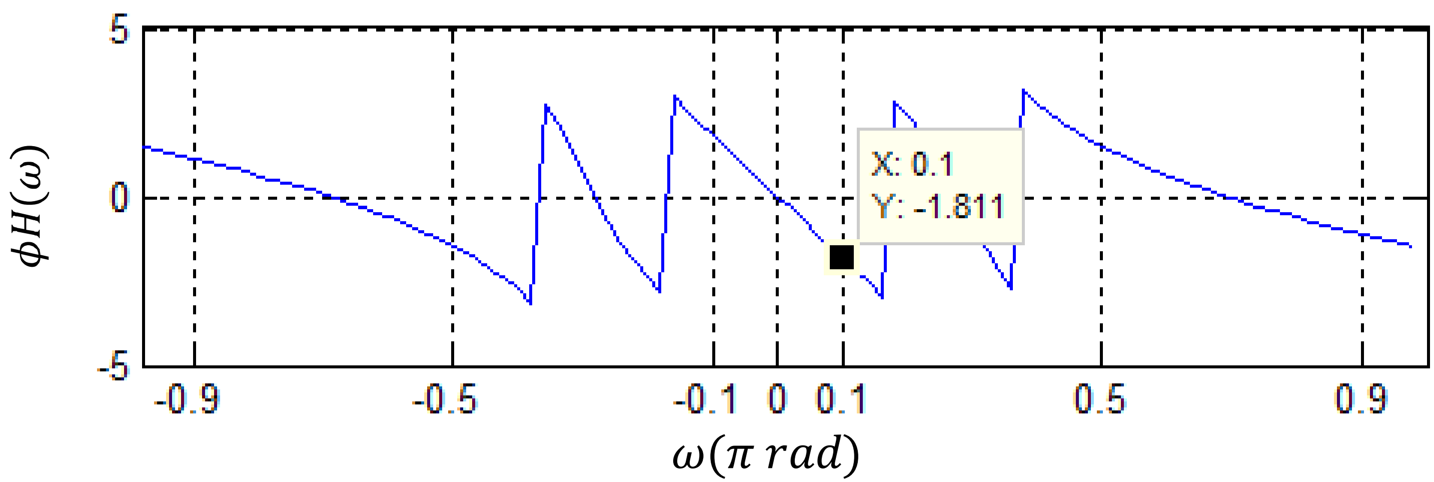
Example 5: 2nd order BF



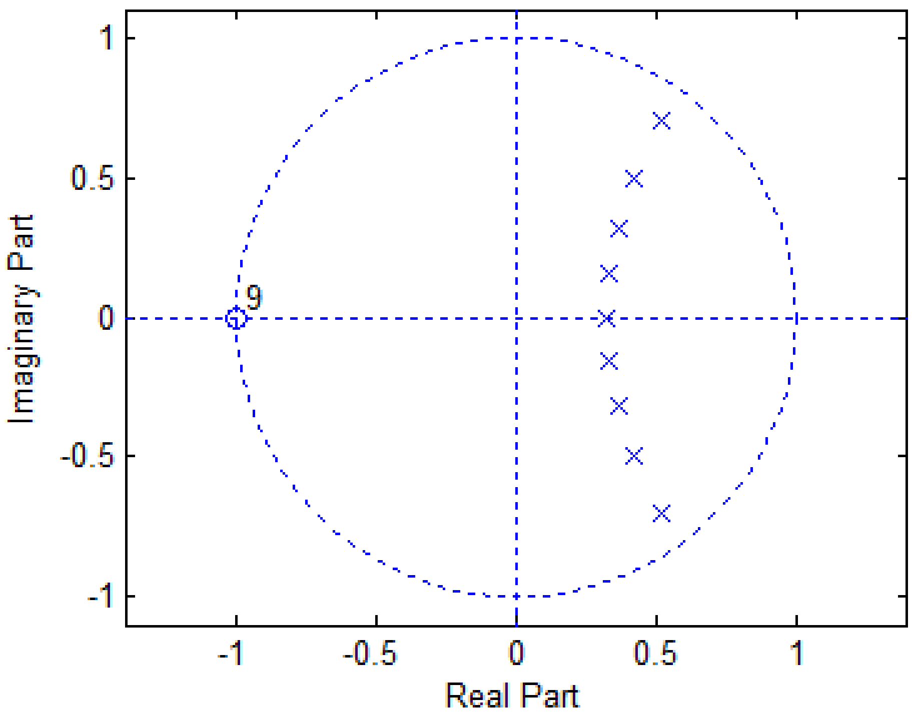
Example 5: 9th order BF

9th order Butterworth filter



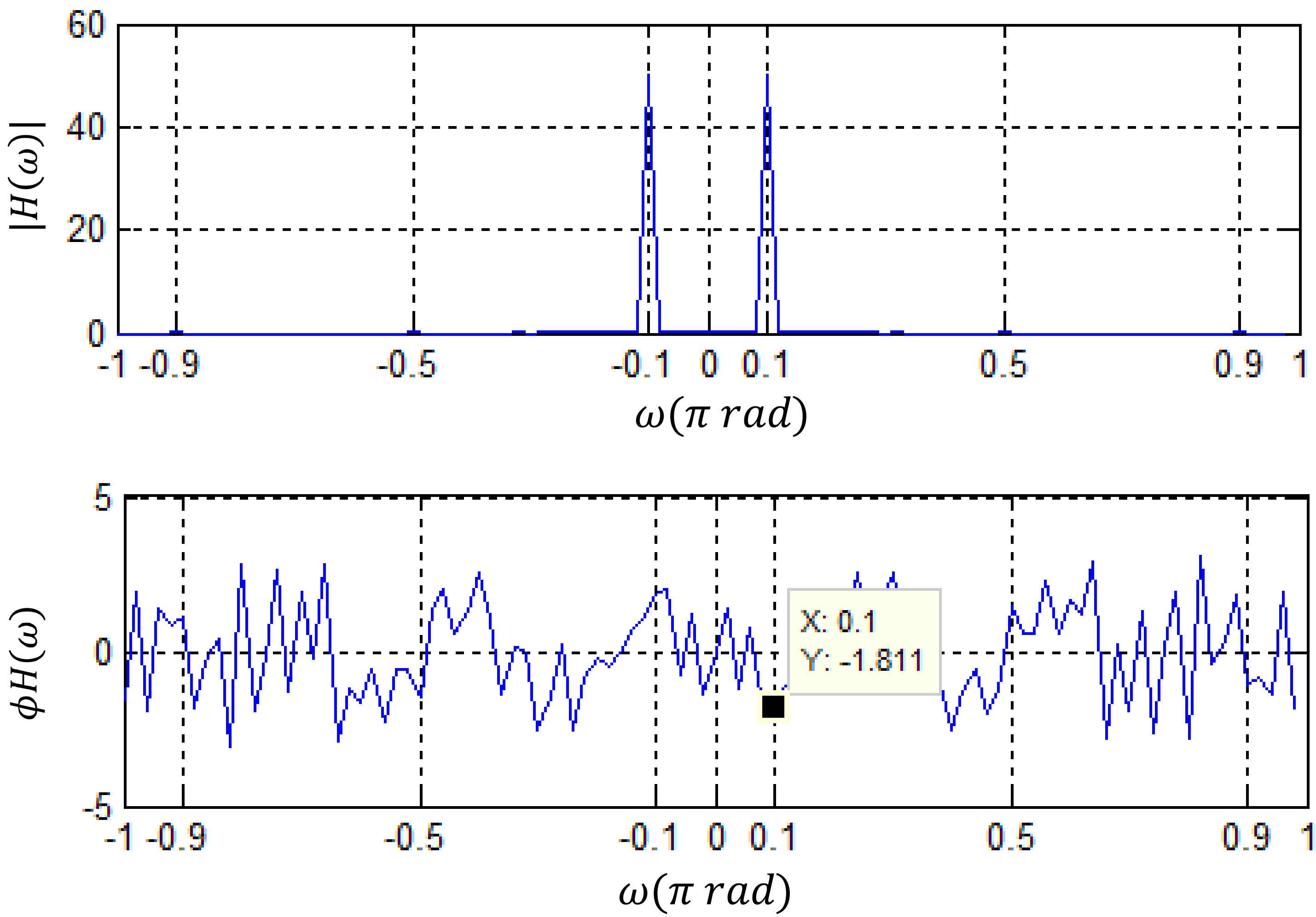


Example 5: 9^{th} order BF



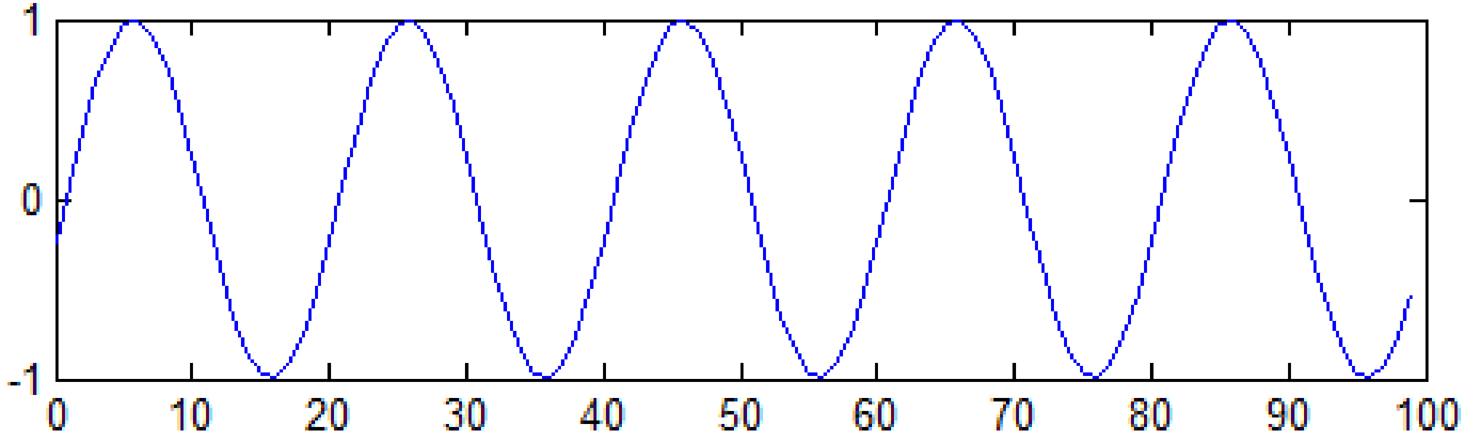
Example 5: 9th order BF

- Output



Example 5: 9^{th} order BF

$y[n]$



n

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