SEL4223 Digital Signal Processing

Discrete Time Fourier Transform

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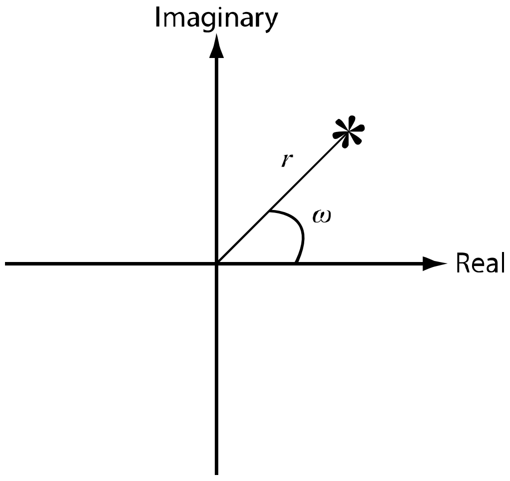
Introduction

- Subset of z-transform
- As z-plane is a complex number plane, $z = re^{j\omega} = a + jb$ where a is the real value, b is the imaginary value. Thus r & ω are as follows

$$r = (a^2 + b^2)^{\frac{1}{2}}$$

$$\omega = \tan^{-1}\left(\frac{b}{a}\right) rad$$



Introduction (cont.)

- In Fourier Transform, $r = 1$. Thus, Fourier Transform is actually z-transform evaluated on the unit circle.
- If the formulation of z-transform is

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

- Formulation for Fourier transform is similar to z-transform but with $z = e^{j\omega}$

Transform Formulation

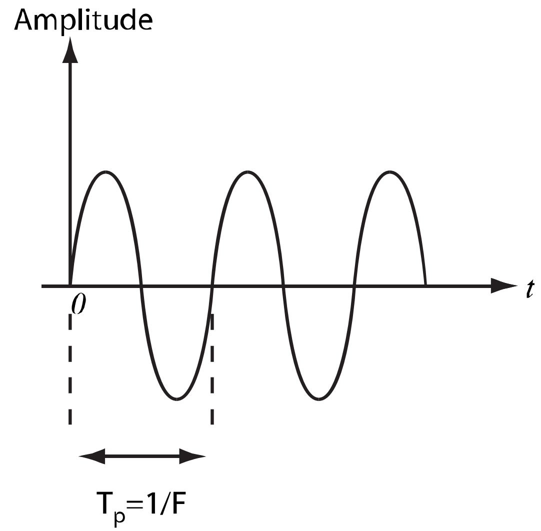
$$Forward \quad \Longrightarrow \quad H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

$$Inverse \quad \Longrightarrow \quad h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

- Note that the inverse FT is evaluated on $\omega = -\pi$ to $\omega = \pi$ or with the range of 2π. This is because ω value will be repeated after each 2π.

Why DTFT?

- Basically, signal has three information, Amplitude (A), Frequency (F) and Phase $(\emptyset)$.



Why DTFT? (cont.)

- In signal filtering system, filtering is based on the frequency of the signal, where the desired frequency component will be preserved while the unwanted frequency component will be removed from the signal.
- In this case, FT is used to sort the signal based on its frequency. When the signal has been sorted based on its frequency, the process of preserving and removing frequency component in the signal will be possible.
- In continuous signal, frequency is labeled with F and the unit is Hz.

Discrete-time frequency

- In discrete signal, frequency is labeled with f where

$$f = \frac{F}{F_s}$$

$F \rightarrow$ frequency of continuous signal – Hz

$F_s \rightarrow$ Sampling frequency – Hz

- As f is the ratio between F and F_s, it has no unit

Discrete-time frequency (cont.)

- Besides Hertz (Hz), frequency is also normally presented in radian as given below, where the frequency is multiplied with 2π

Continuous signal $\rightarrow \Omega = 2\pi F \quad (rads^{-1})$

Discrete signal $\rightarrow \omega = 2\pi f \quad (rad)$

- Based on the FT formulation, it can be seen that the FT is sorting the frequency component of the signal based on ω

Discrete-time frequency (cont.)

- As the important component of the FT is ω, sometimes the formulation is written as $H(\omega)$ instead of $H(e^{j\omega})$.
- If we want to present the FT in terms of f, the formulation becomes

$$H(f) = \sum_{n=-\infty}^{\infty} h[n]e^{-j2\pi fn}$$

Notation

	Time-domain	Frequency-domain
Continuous-time	$x(t)$ Function (signal) – small letter Index (time) – small letter	$X(\Omega), X(F)$ Function (response) – Capital letter Index (frequency) – Capital letter
Discrete-time	$x[n]$ Function (signal) – small letter Index (integer) – small letter	$X(\omega), X(f)$ Function (response) – Capital letter Index (frequency) – small letter

Magnitude and Phase Spectrum

- Similar to z-transform, results of FT is also a complex value where the real and imaginary value is separated.
- Thus, to obtain the behavior of the signal, magnitude and phase spectrum are used for analysis

$$H(\omega) = H_R(\omega) + jH_I(\omega)$$

Magnitude $\rightarrow |H(\omega)| = \sqrt{(H_R(\omega))^2 + (H_I(\omega))^2}$

Phase $\rightarrow \angle H(\omega) = \tan^{-1} \frac{H_I(\omega)}{H_R(\omega)}$

Other Spectrum Representation

- In signal analysis, few other spectrums are also used. There are;

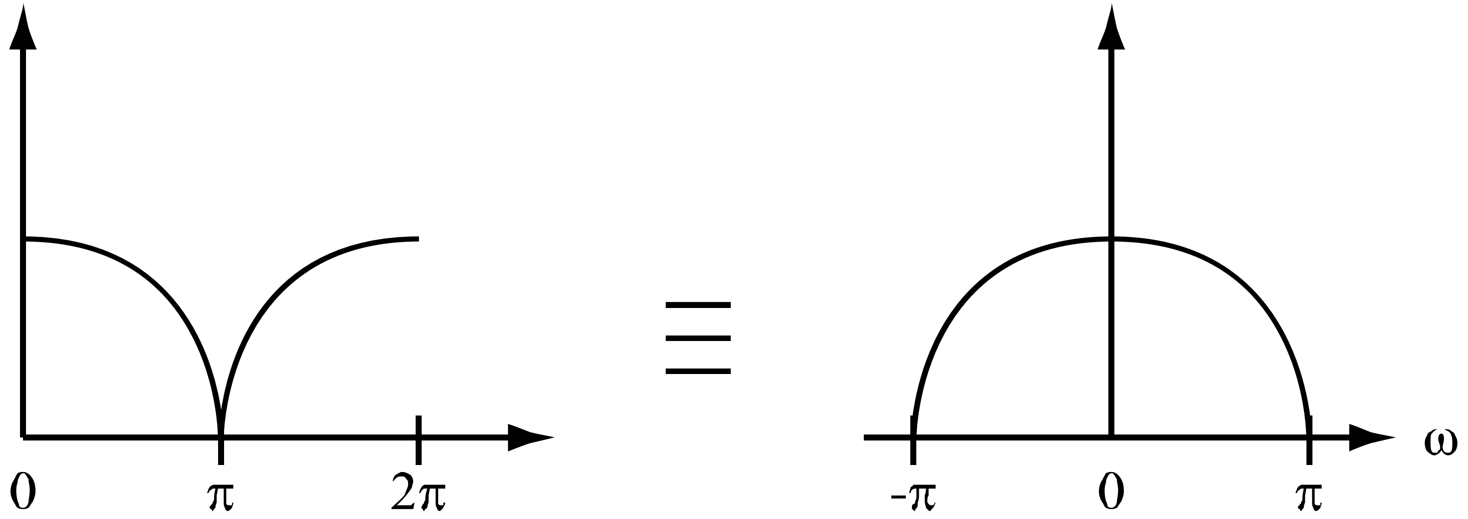
  Energy $\rightarrow |H(\omega)|^2 = H(\omega)H^*(\omega)$

  Magnitude dB $\rightarrow 20\log_{10}|H(\omega)|$
  (To obtain clearer plots as most of $|H(\omega)|$ values are small)

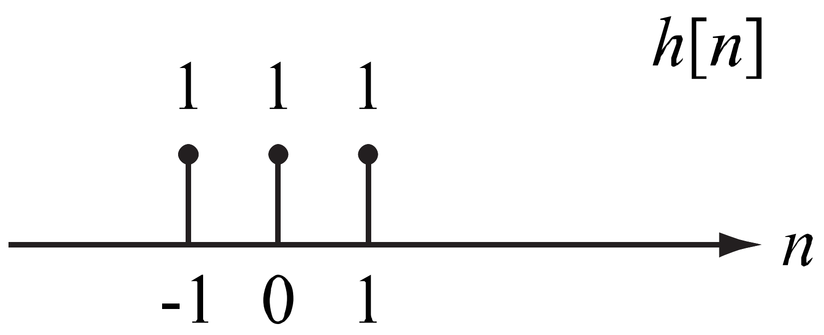
- Obviously, x-axis of all of the spectrums ($|H(\omega)|, \angle H(\omega), |H(\omega)|^2, |H(\omega)|_{dx}$) is the frequency component of signal, ω and y-axis of the spectrums is the value of the spectrum.

DTFT Plot

- Normally, the spectrums will be plotted from $\omega = -\pi$ to $\omega = \pi$.
- As the spectrum value will be repeated every 2π, an alternative way of plotting the spectrum is from $\omega = 0$ to $\omega = 2\pi$.



Example 1



- Plot $|H(\omega)|, \angle H(\omega), |H(\omega)|^2$

Solution:

- $h[n] = \delta[n+1] + \delta[n] + \delta[n-1]$

Example 1 (cont.)

$$H(\omega) = \sum_{n=-\infty}^{\infty} (\delta[n+1] + \delta[n] + \delta[n-1])e^{-j\omega n}$$

$$= e^{j\omega} + 1 + e^{-j\omega}$$

$$= 1 + e^{j\omega} + e^{-j\omega}$$

$$= 1 + 2\cos(\omega)$$

$$H(\omega) = 1 + 2\cos(\omega)$$

$$H_R(\omega) = 1 + 2\cos(\omega)$$

$$H_I(\omega) = 0$$

Trigonometry Equations

In trigonometry, the relationship between cosine and sine to exponential is as below
$cos(\omega) = \dfrac{e^{j\omega} + e^{-j\omega}}{2}$
$sin(\omega) = \dfrac{e^{j\omega} - e^{-j\omega}}{j2}$
$e^{j\omega} = cos(\omega) + jsin(\omega)$

Example 1 (cont.)

- $|H(\omega)| = \sqrt{(H_R(\omega))^2 + (H_I(\omega))^2}$

  $= |1 + 2\cos(\omega)|$

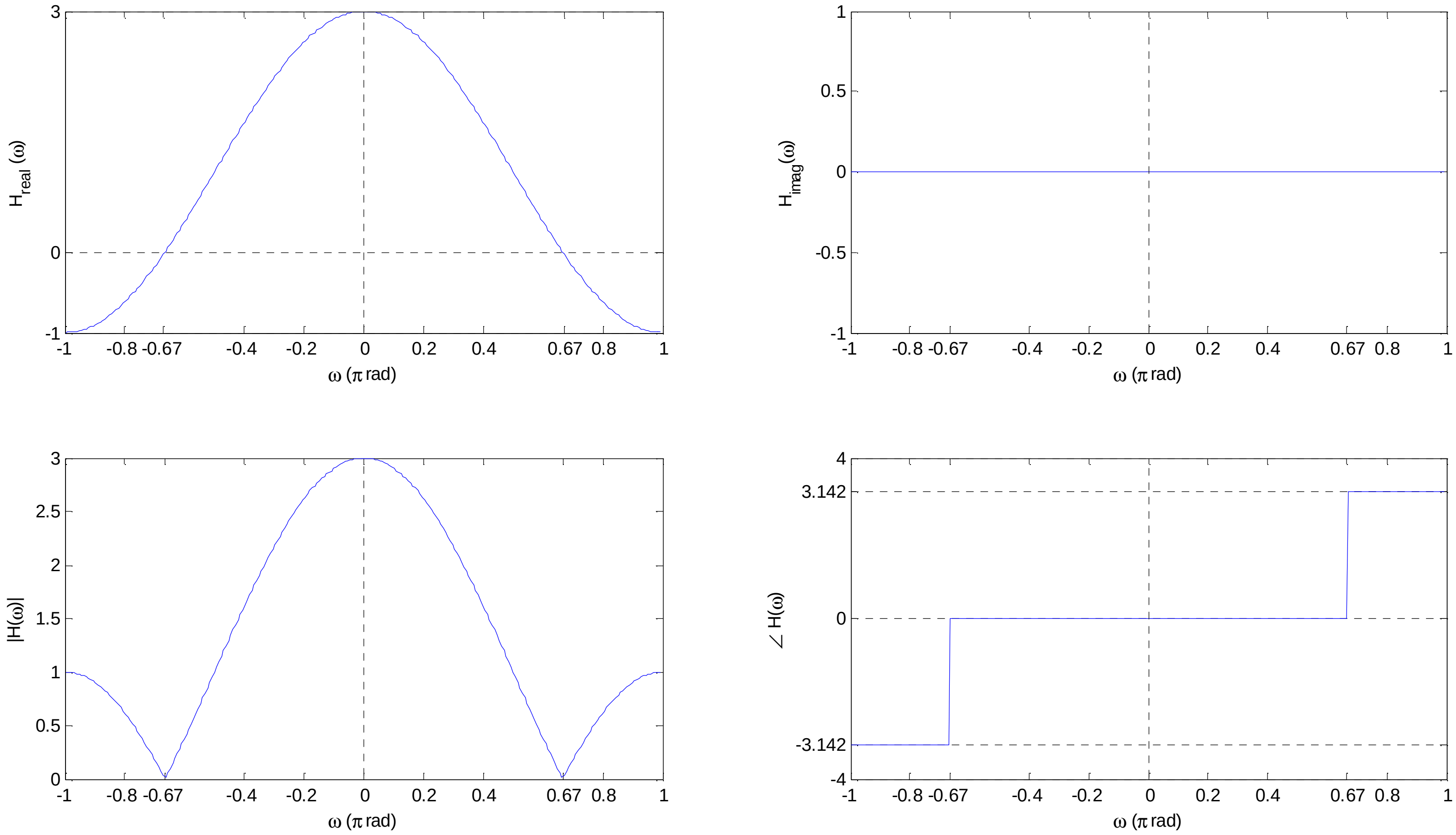
- $\angle H(\omega) = \tan^{-1}\frac{H_I(\omega)}{H_R(\omega)}$

  $= \tan^{-1}\frac{0}{1+2\cos(\omega)}$

  $= \begin{cases} 0 \; if \; H(\omega) \geq 0 \\ \pi \; if \; H(\omega) < 0 \\ -\pi \; if \; H(-\omega) < 0 \end{cases}$

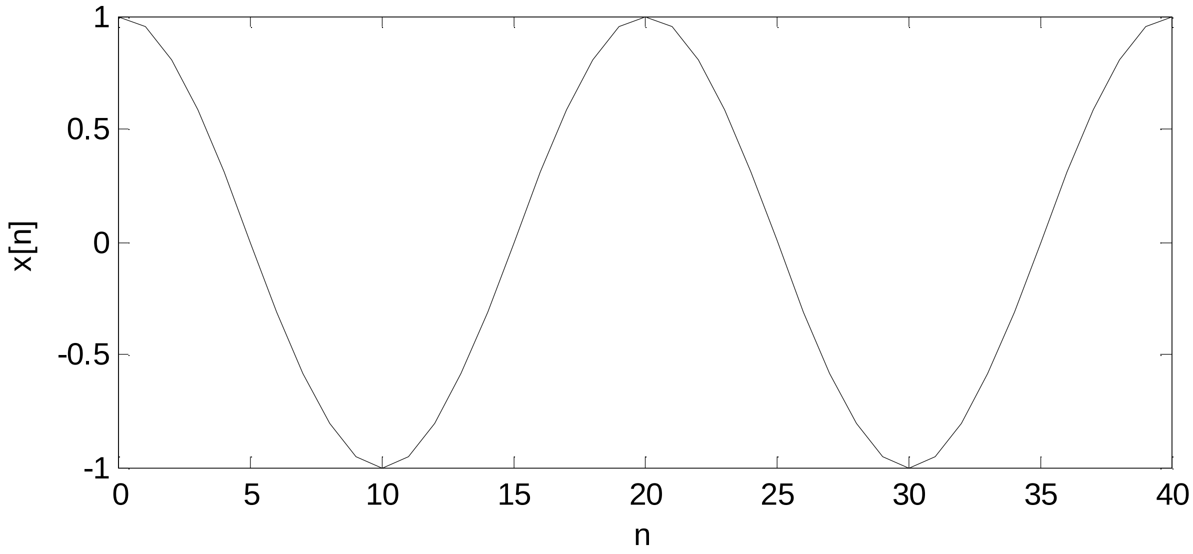
- $|H(\omega)|^2 = (1 + 2\cos(\omega))^2$

Example 1 (cont.)



Example 2

- $x[n] = \cos(0.1\pi n)$ for $0 \leq n \leq 40$



- From the equation above, it is known that $f = 0.05$ where $\omega = 2\pi f = 0.1\pi$. The frequency value can also be seen from figure above where $f = \frac{1}{T_p} = \frac{1}{20} = 0.05$. Now, lets plot the signal in the frequency domain using Fourier Transform.

Example 2 (cont.)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$= \sum_{n=0}^{40} \cos(0.1\pi n)e^{-j\omega n}$$

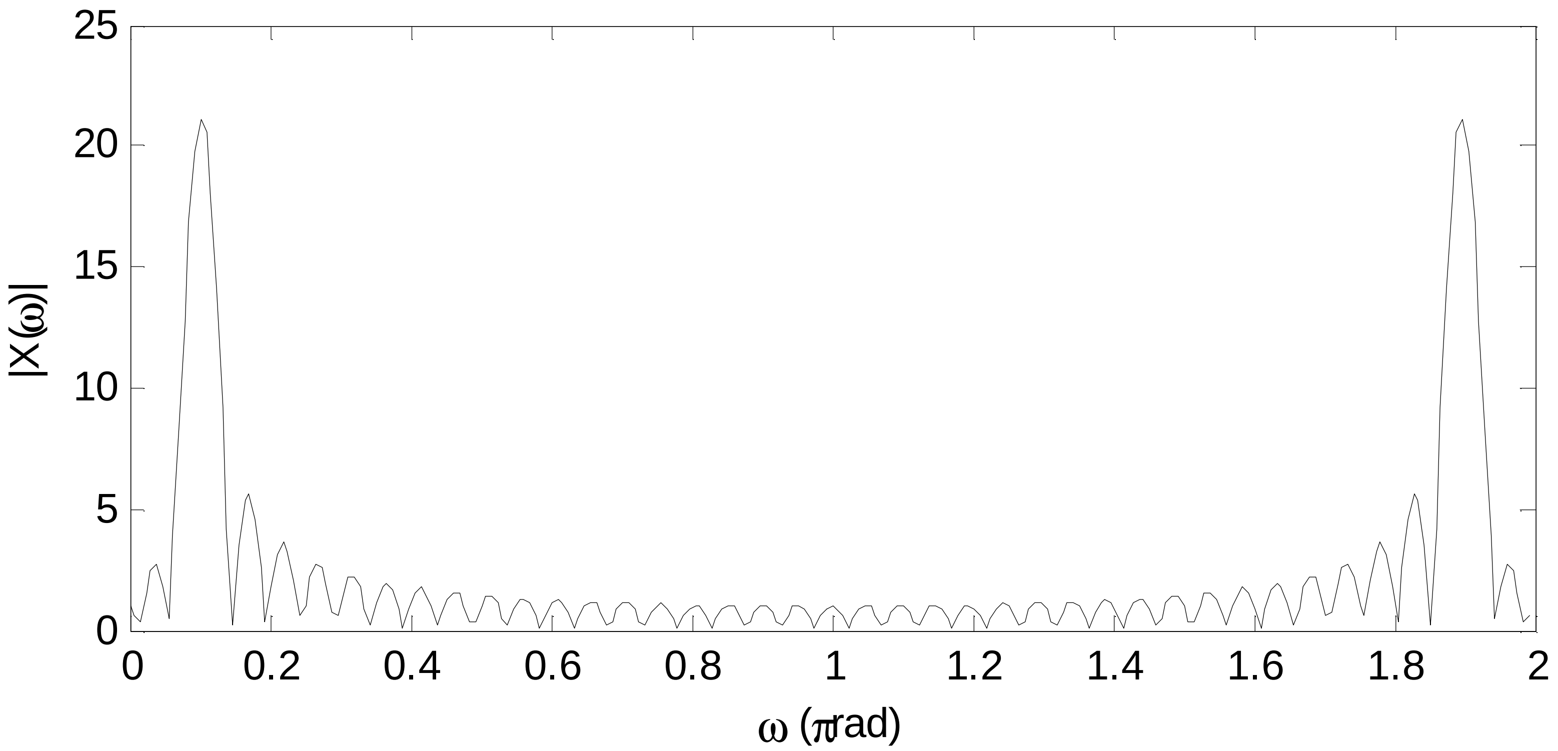
$$= \sum_{n=0}^{40} \frac{1}{2}\left(e^{j0.1\pi n} + e^{-j0.1\pi n}\right)e^{-j\omega n}$$

$$= \frac{1}{2}\sum_{n=0}^{40}\left(e^{-j(\omega-0.1\pi)n} + e^{-j(\omega+0.1\pi)n}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{40} e^{-j(\omega-0.1\pi)n} + \frac{1}{2}\sum_{n=0}^{40} +e^{-j(\omega+0.1\pi)n}$$

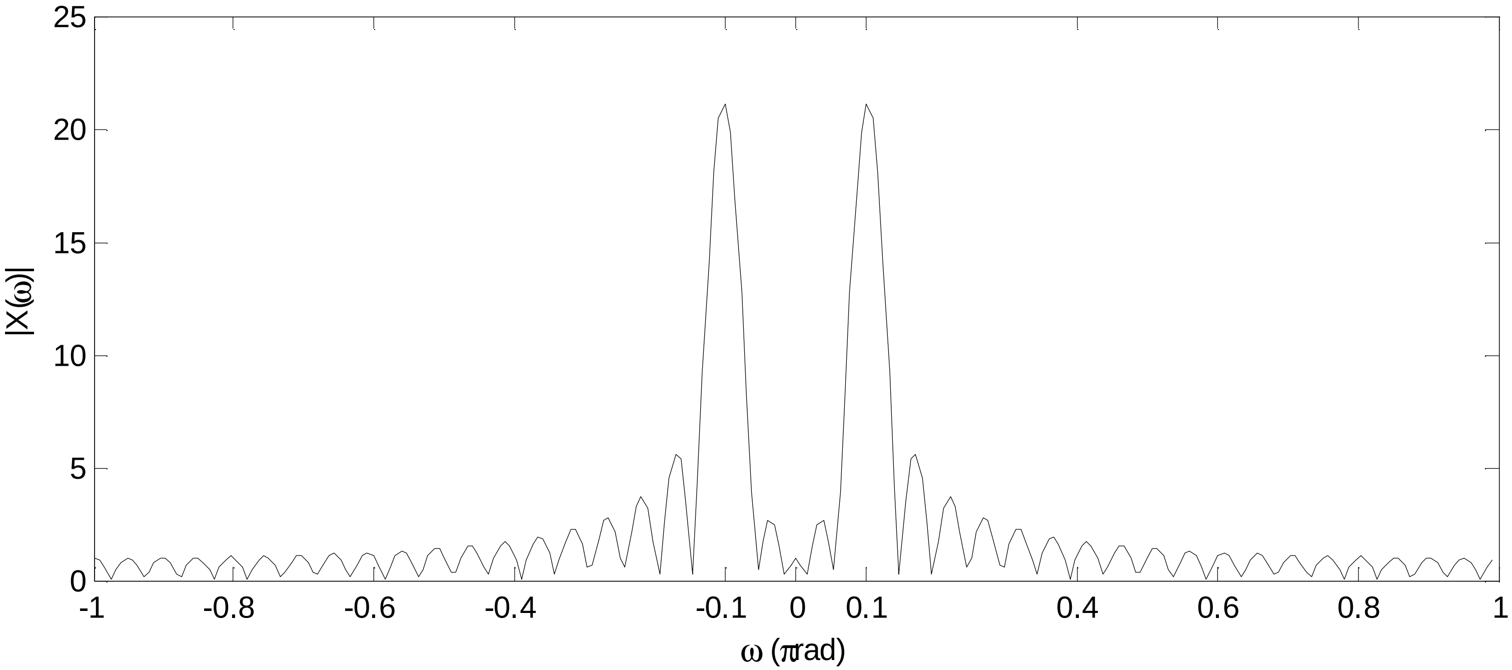
$$= \frac{1}{2}\left(\frac{1-e^{-j41(\omega-0.1\pi)}}{1-e^{-j(\omega-0.1\pi)}} + \frac{1-e^{-j41(\omega+0.1\pi)}}{1-e^{-j(\omega+0.1\pi)}}\right)$$

Example 2 (cont.)



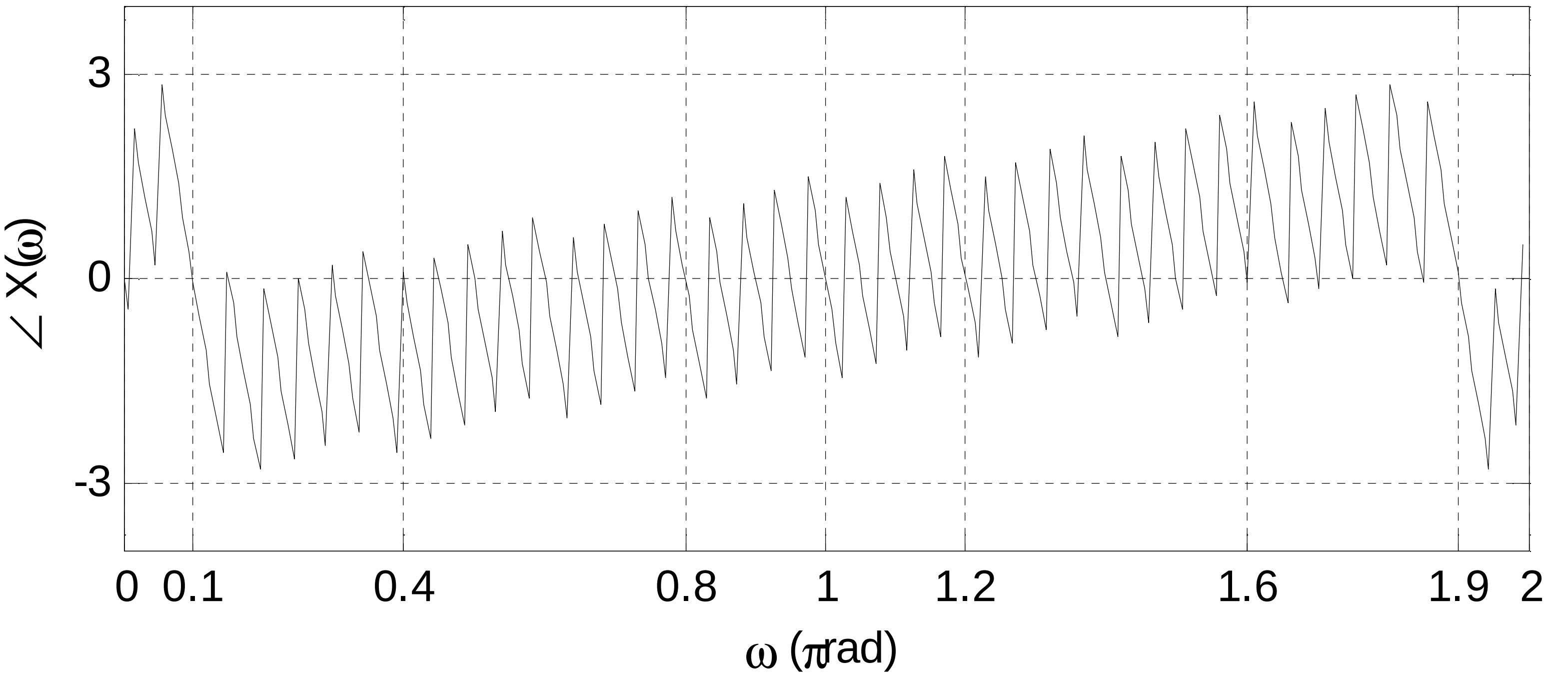
Magnitude spectrum where $0 \leq \omega \leq 2\pi$

Example 2 (cont.)



Magnitude spectrum where $-\pi \leq \omega \leq \pi$

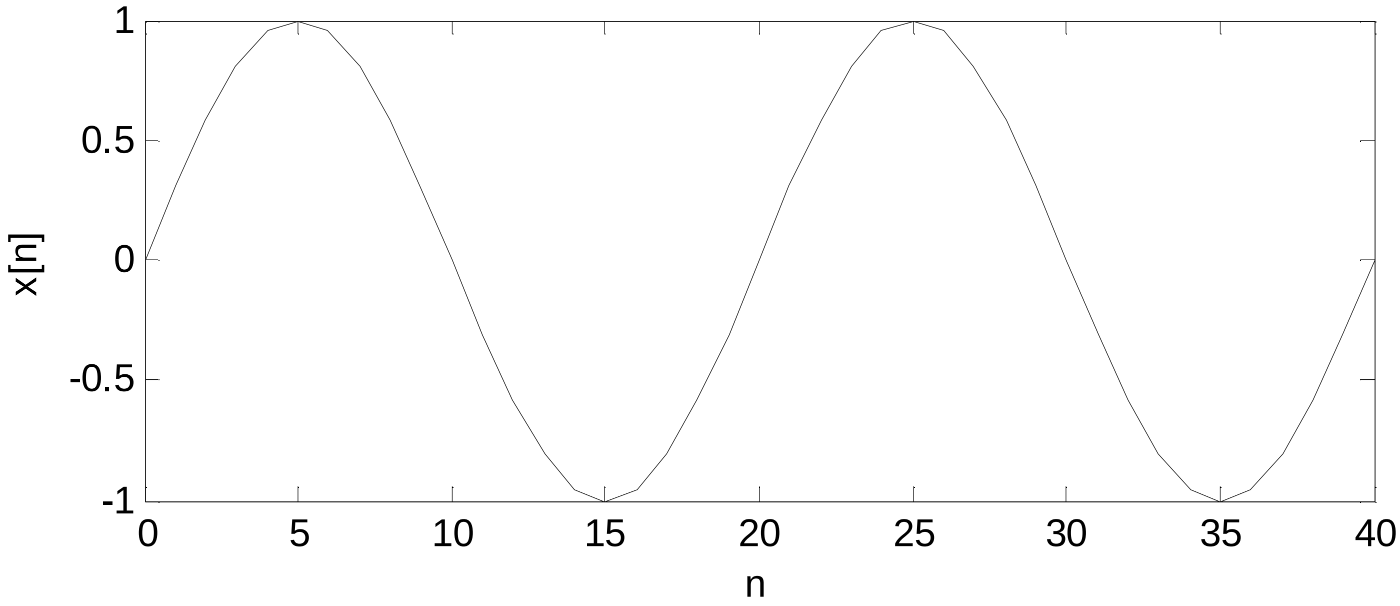
Example 2 (cont.)



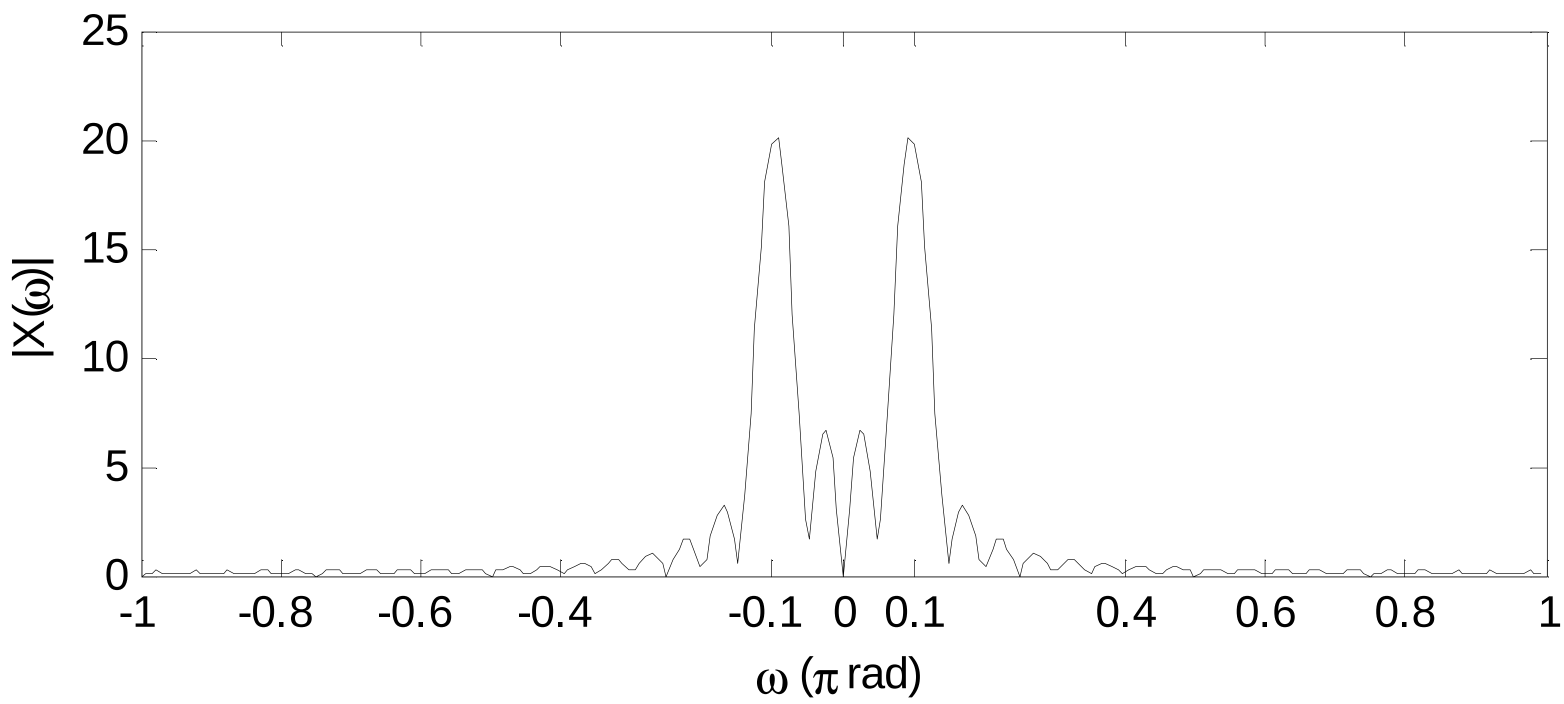
Phase spectrum where $0 \leq \omega \leq 2\pi$

Example 3

- $x[n] = \cos(0.1\pi n - 0.5\pi)$ for $0 < n < 40$
- Figure below shows $x[n]$ where it is a 5 samples delay $(phase = -0.5\pi)$ of signal in Figure 5.18

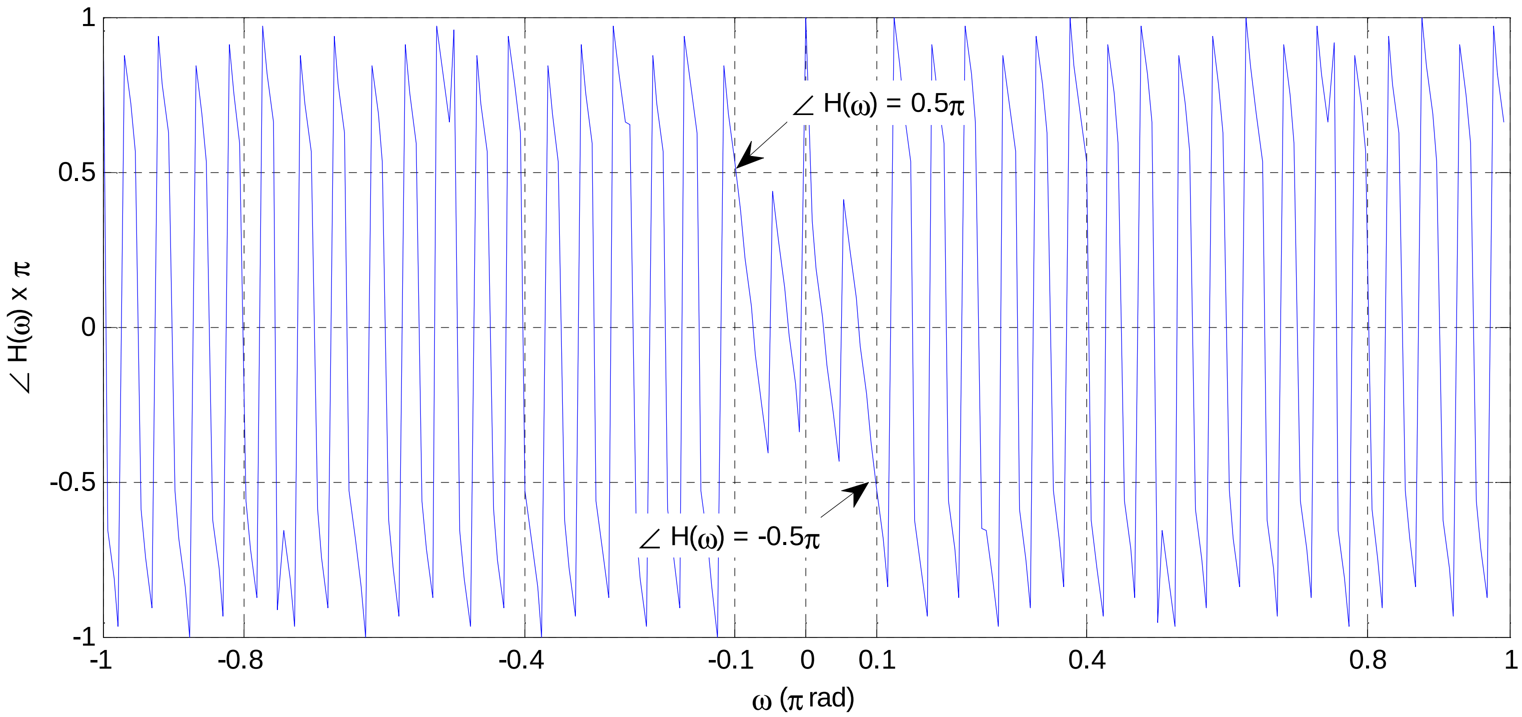


Example 3 (cont.)



Magnitude Spectrum

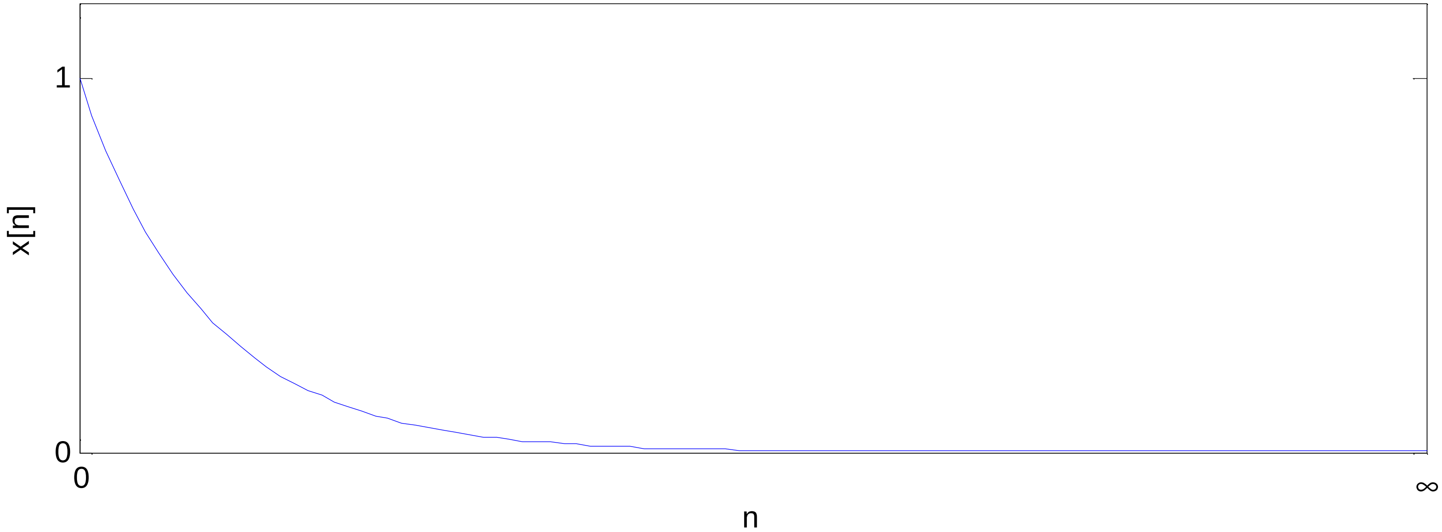
Example 3 (cont.)



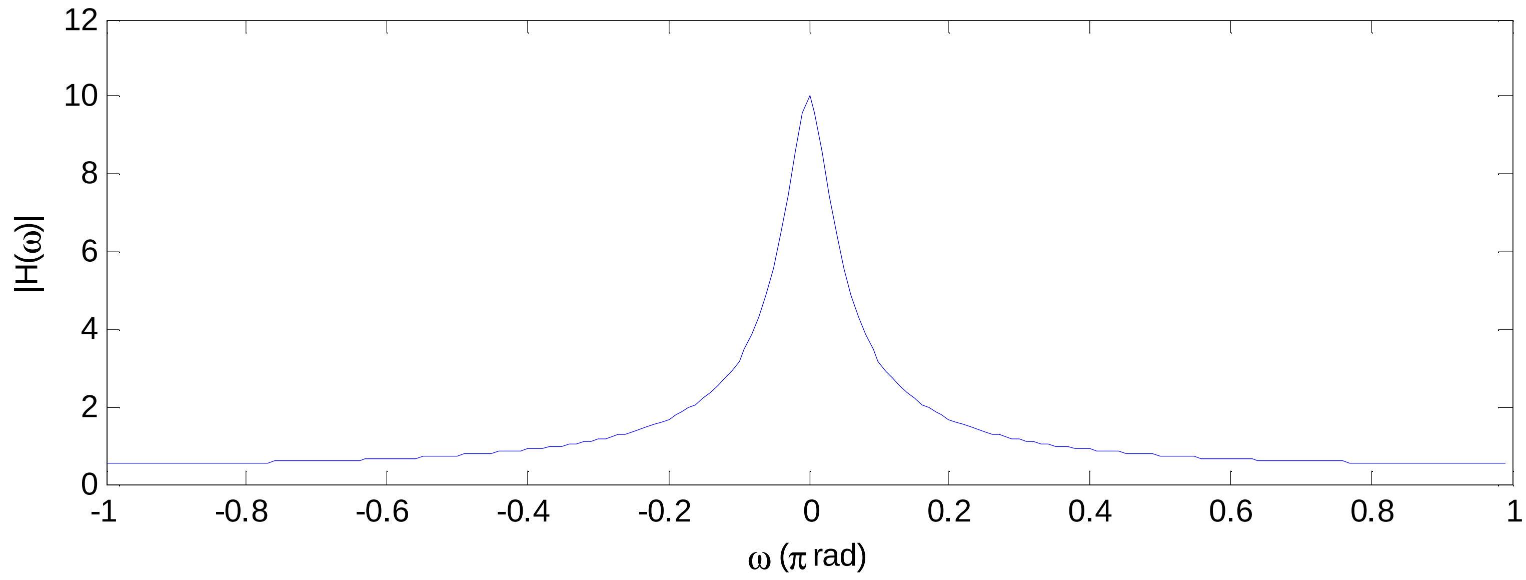
Phase spectrum. The arrows show the phase value at $\omega = 0.1\pi$

Example 4

- $x[n] = 0.9^n u[n]$
- $X(\omega) = \frac{1}{1-0.9e^{-j\omega}}$

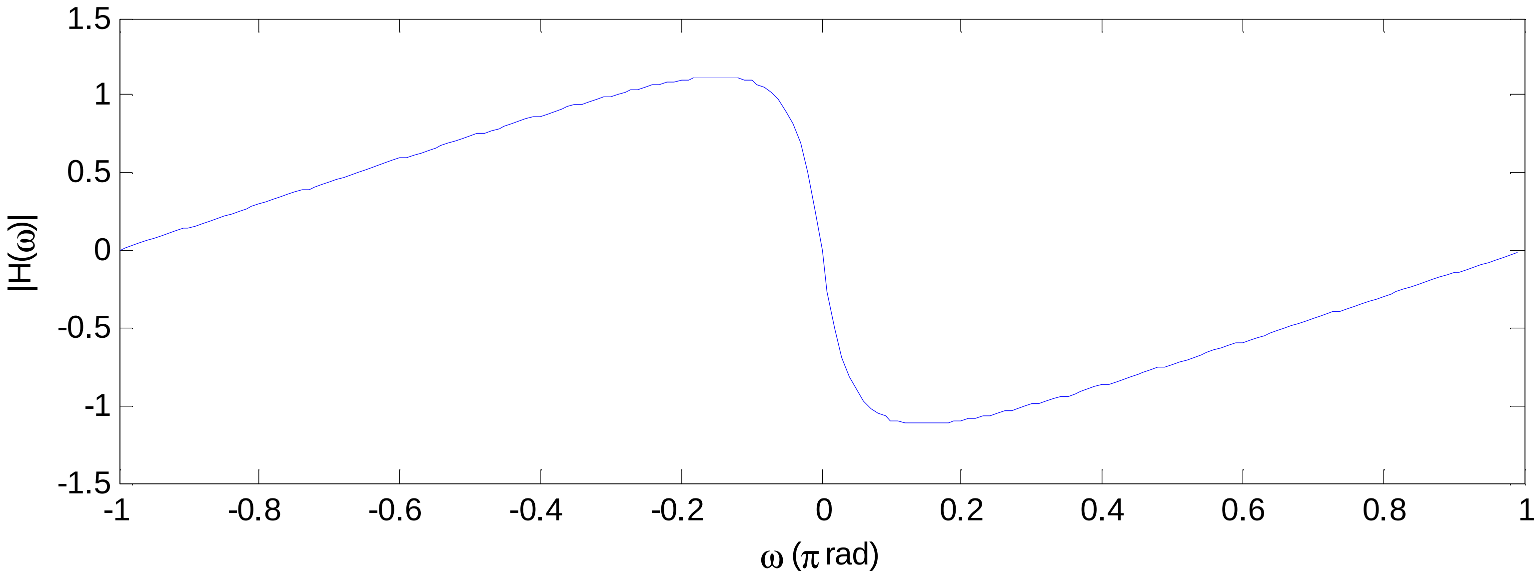


Example 4 (cont.)



Magnitude Spectrum

Example 4 (cont.)



Phase Spectrum

References

1) John G. Proakis, Dimitris K Manolakis, "Digital Signal Processing: Principle, Algorithm and Applications", Prentice-Hall, 4th edition (2006).
2) Sanjit K. Mitra, "Digital Signal Processing-A Computer Based Approach", McGraw-Hill Companies, 3rd edition (2005).
3) Alan V. Oppenheim, Ronald W. Schafer, "Discrete-Time Signal Processing", Prentice-Hall, 3rd edition (2009).