SEL4223 Digital Signal Processing

Spectrum representation of discrete-time signals

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Fourier transformation family

		Periodic Signal	Aperiodic Signal
Continuous-time signal	Continuous frequency	-	Fourier Transform (FT) $X(\Omega), X(F)$
	Discrete frequency	Fourier Series (FS) a_k	-
Discrete-time signal	Continuous frequency	-	Discrete-time Fourier Transform (DTFT) $X(\omega), X(f)$
	Discrete frequency	Discrete-time Fourier Series (DTFS) a_k	Discrete Fourier Transform (DFT) Fast Fourier Transform (FFT) $X[k]$

Fourier Series

- Fourier series is the foundation to all of the Fourier transformation family
- The name Fourier Series is actually refers to a representation of periodic signal in terms of sine and cosine signal where sine and cosine signals are known to be a single frequency signal.
- In other words, Fourier Series decomposes periodic signal into a series of single frequency signals.

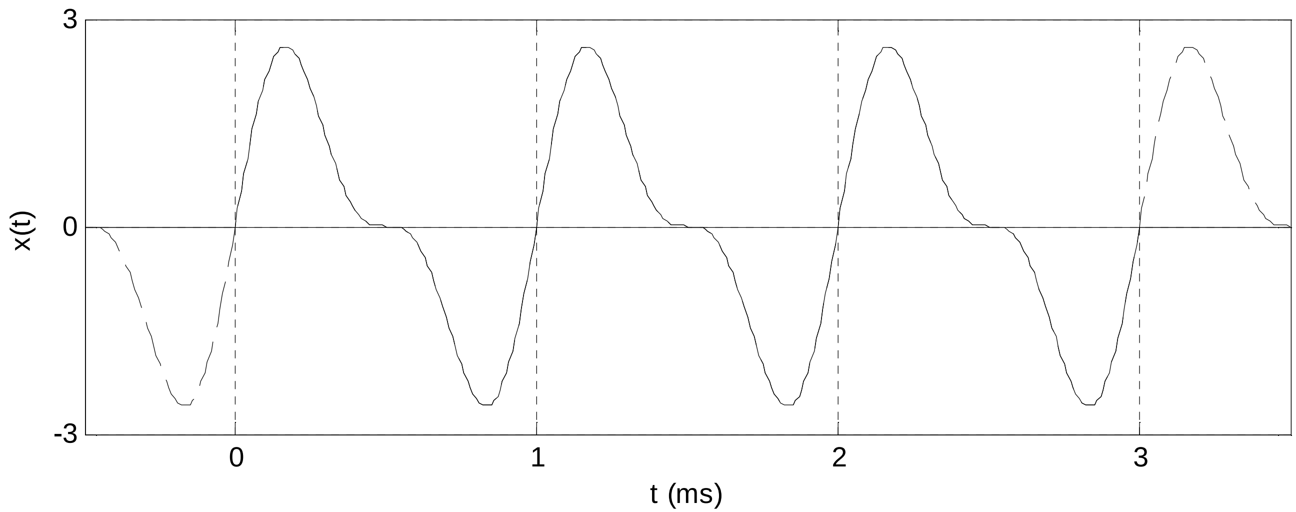
Fourier Series (cont.)

- Frequency response of the Fourier Series is called ‘Fourier Series Coefficient’.

$$\begin{matrix}\text{Fourier series, } x(t) \\ (\text{time} - \text{domain})\end{matrix} \xrightarrow{\text{analysis}} \begin{matrix}\text{Fourier Series Coefficient, } a_k \\ (\text{frequency} - \text{domain})\end{matrix}$$

Example 1

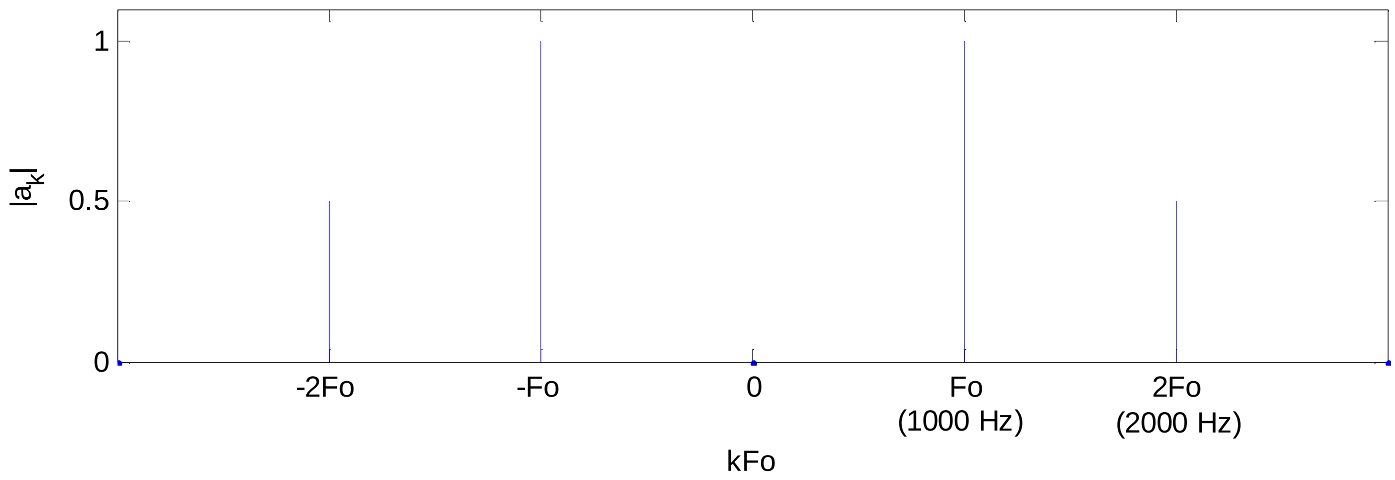
- Periodic signal $x(t) = 2\sin(2\pi(1000)t) + \sin(2\pi(2000)t)$



- The time period of the signal is $\mathrm{T_p} = 1\ \mathrm{ms}$.
- Frequency based on the $\mathrm{T_p}$ is called fundamental frequency, $\mathrm{F_o}$. In this case, $\mathrm{F_o} = 1000\ \mathrm{Hz}$.

Example 1 (cont.)

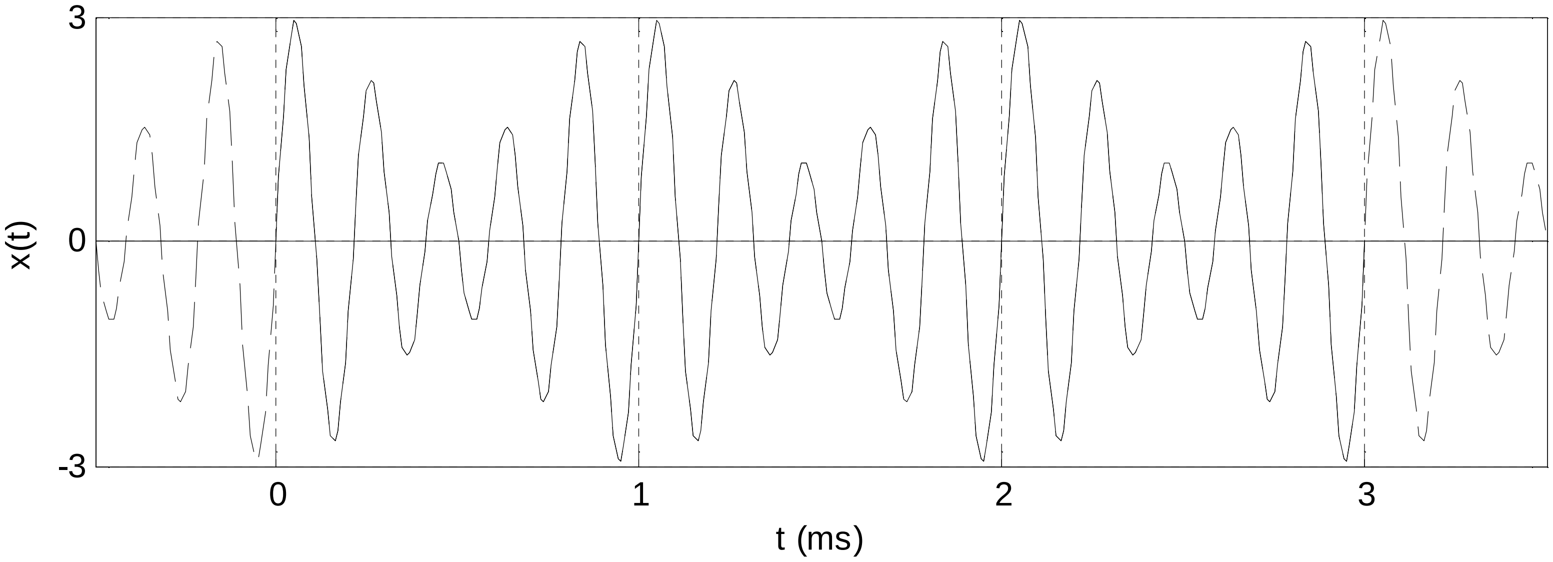
- Thus, F_o is actually the lowest frequency that can occur in the periodic signal. Other frequencies exist in the periodic signal can be represented as multiple of F_o



- From above figure, it can be seen that each of the frequencies component contain in the periodic signal is represented with the Fourier Series coefficient value where $|a_k|$ is half of the amplitude of the signal for that frequency.

Example 2

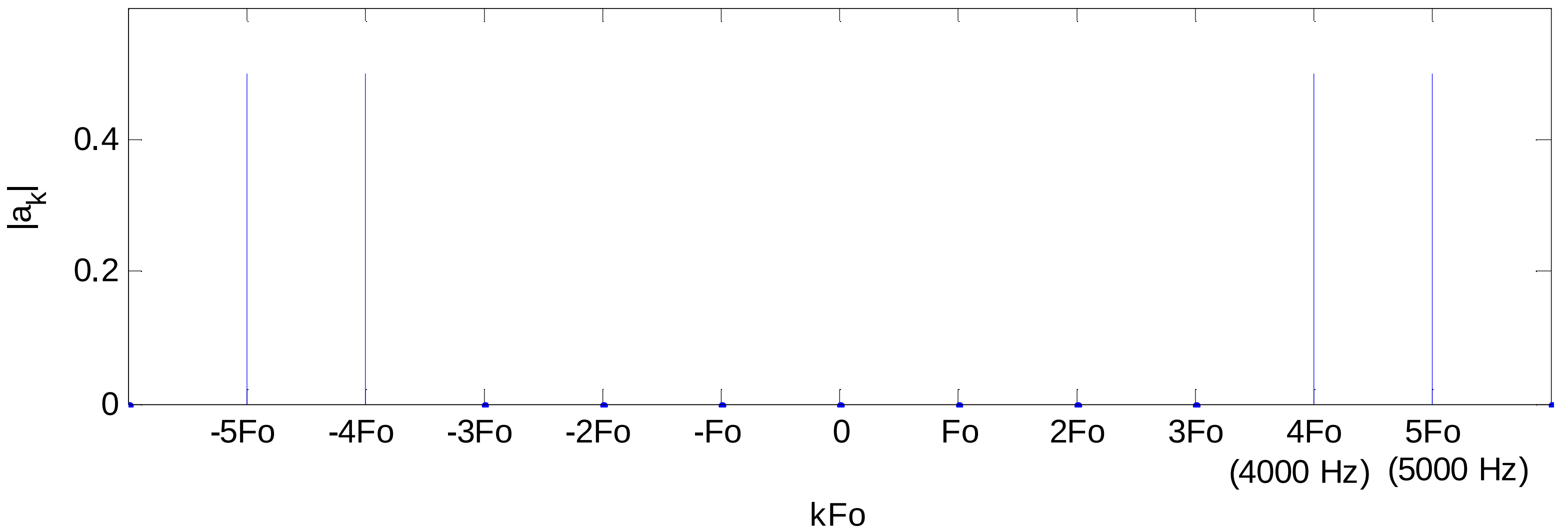
- Periodic signal $x(t) = \sin(2\pi(4000)t) + \sin(2\pi(5000)t)$



- As $\mathrm{T_p} = 1\ \mathrm{ms}$, $F_o = 1000Hz$

Example 2 (cont.)

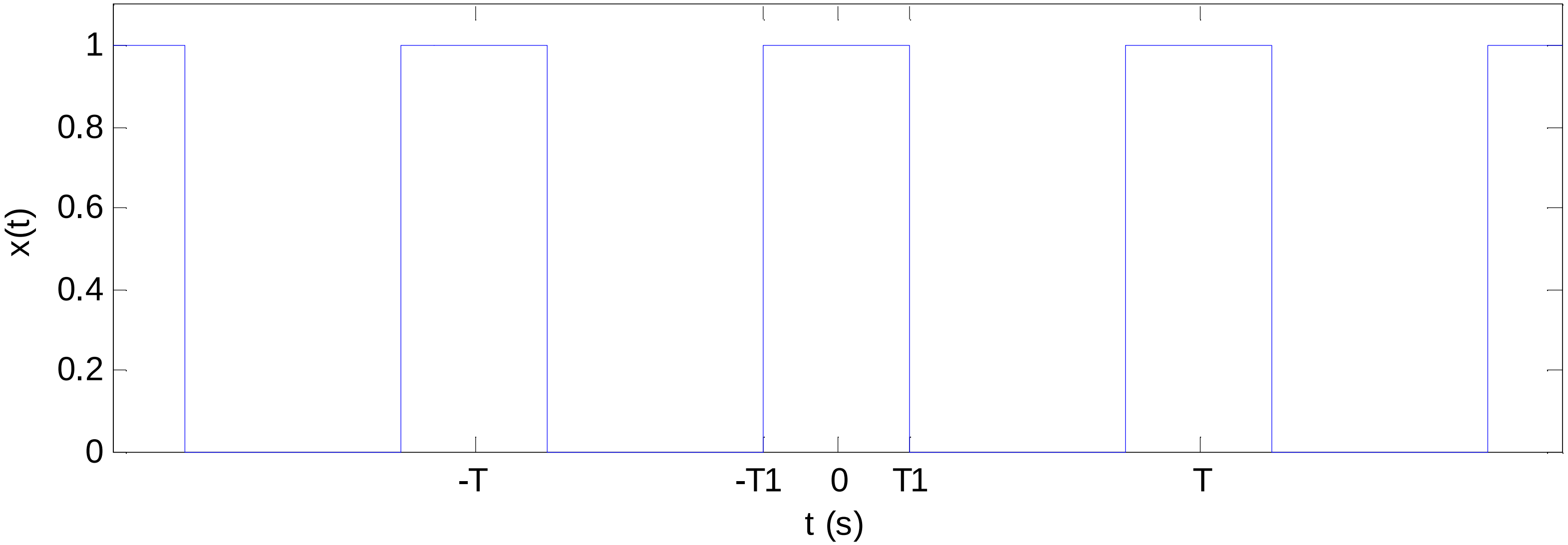
- Frequency response of the signal is shown below



- Because frequency for Fourier Series is represented as multiple of F_o, normally the frequency response plot ignore the F_o and represent the frequency only with k, which is an integer number. This is why Fourier Series is said to have discrete frequency.

Example 3

- Square wave: $x(t) = \begin{cases} 1 & \text{for } |t| < T1 \\ 0 & \text{for } T1 < |t| < \frac{T}{2} \end{cases}$



Example 3 (cont.)

- Fourier Series Coefficient:

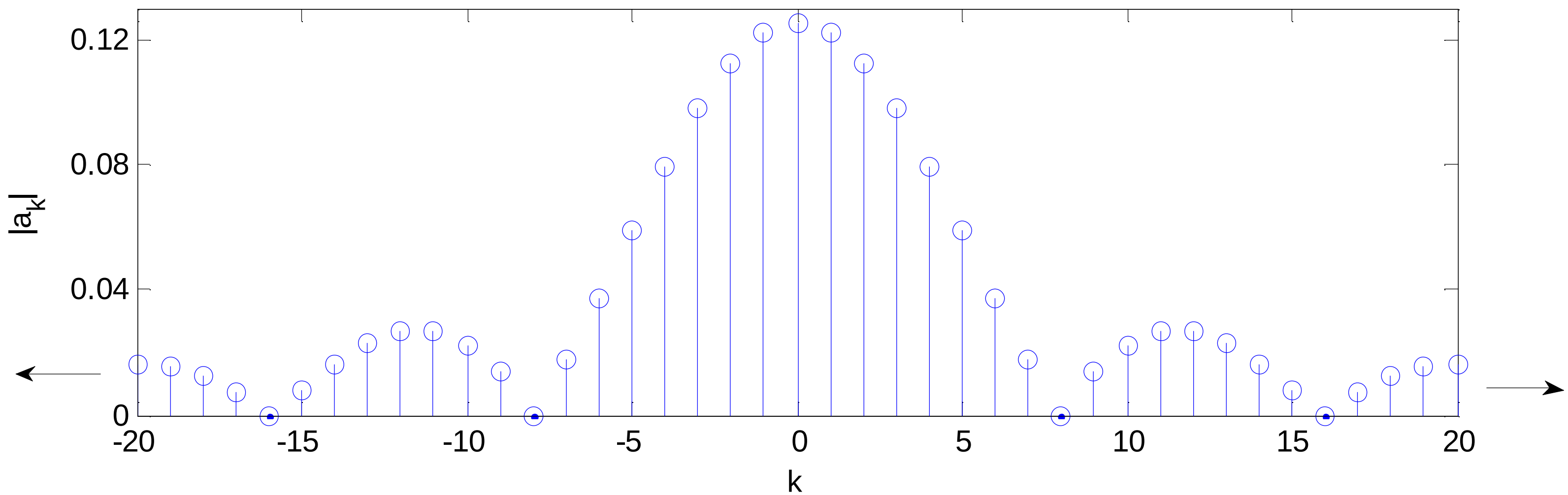
$$a_k = \begin{cases} \frac{2T1}{T} & \text{for } k = 0 \\ \frac{\sin\left(2\pi k \frac{T1}{T}\right)}{k\pi} & \text{for } k \neq 0 \end{cases}$$

- Fourier series:

$$x(t) = \frac{4}{\pi} \sum_{k=1,3,5,\ldots}^{\infty} \frac{1}{k} \sin\left(\frac{2\pi k t}{T}\right)$$

- Frequency response of the signal is shown on the following figure

Example 3 (cont.)



- If $T = 1$ ms, $F_o = 1000$ Hz and each increment of k on above figure will have an increase of 1000 Hz
- k value in the figure is not ends at $k = 20$, but will continue until $k = \infty$ and $k = -\infty$. This means that constructing a periodic square wave signal needs an infinity number of frequency component.

Fourier Transform

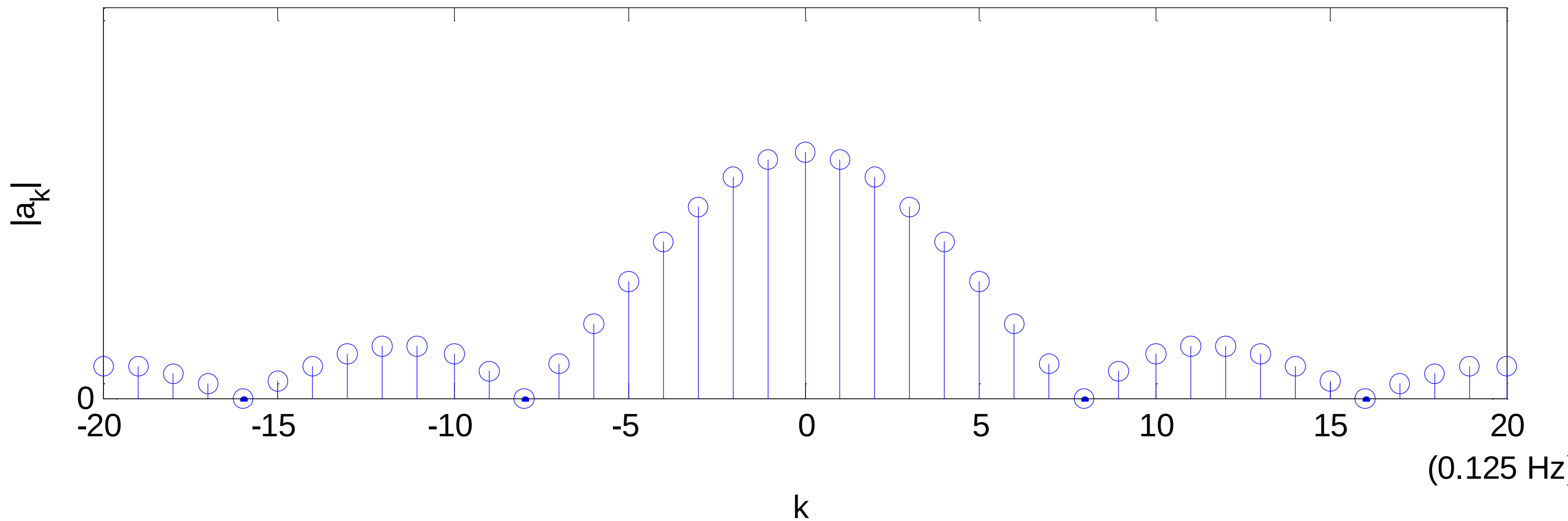
- Fourier transform is use to obtain frequency response of an aperiodic signal.
- Aperiodic signal can be assume to be periodic by setting $T_p = \infty$.
- Thus, Fourier series formulation can be used where the fundamental frequency is $F_o = \frac{1}{\infty} \approx 0$.
- Because $F_o = 0$, there are no frequency gap between the k value in the frequency response. In Example 1 and Example 2, the frequency gap is $1000\ Hz$.

Fourier Transform (cont.)

- Hence:
  - Frequency for Fourier Transform is in continuous form
  - Frequency for Fourier Series is in discrete form.
- In other words:
  - Aperiodic signal contains all frequency values
  - Periodic signal only contains frequency at multiple value of its fundamental frequency.

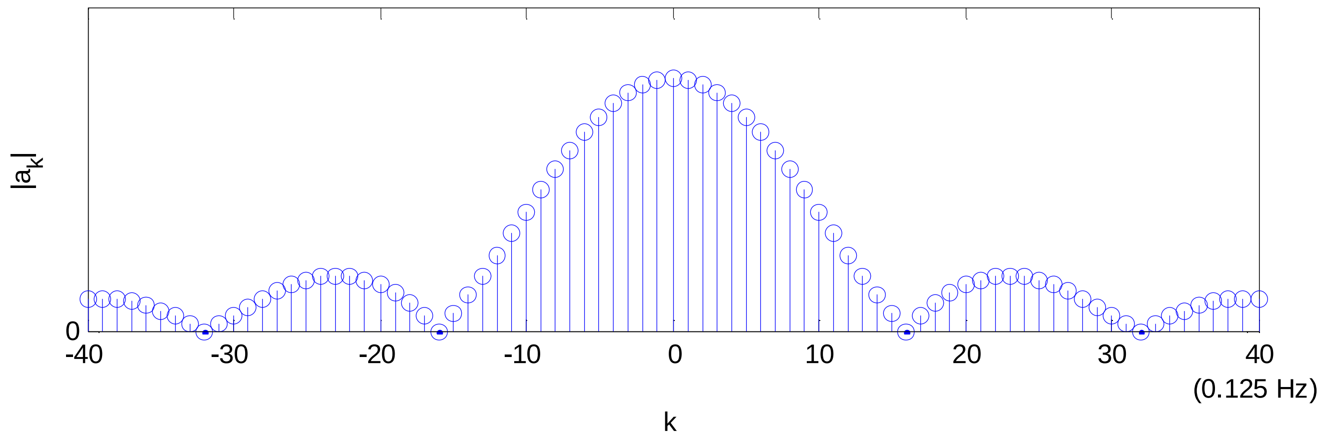
Example 4

- Let's repeat Example 3 with few different T values and constant $\mathrm{T1} = 10s$.

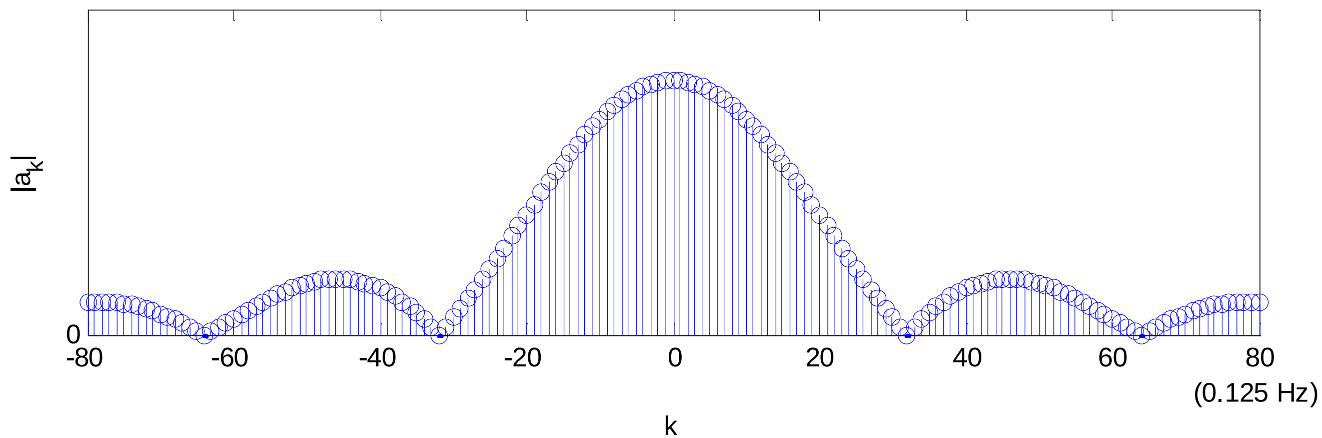


Frequency response for $\mathrm{T} = 160s$

Example 4 (cont.)

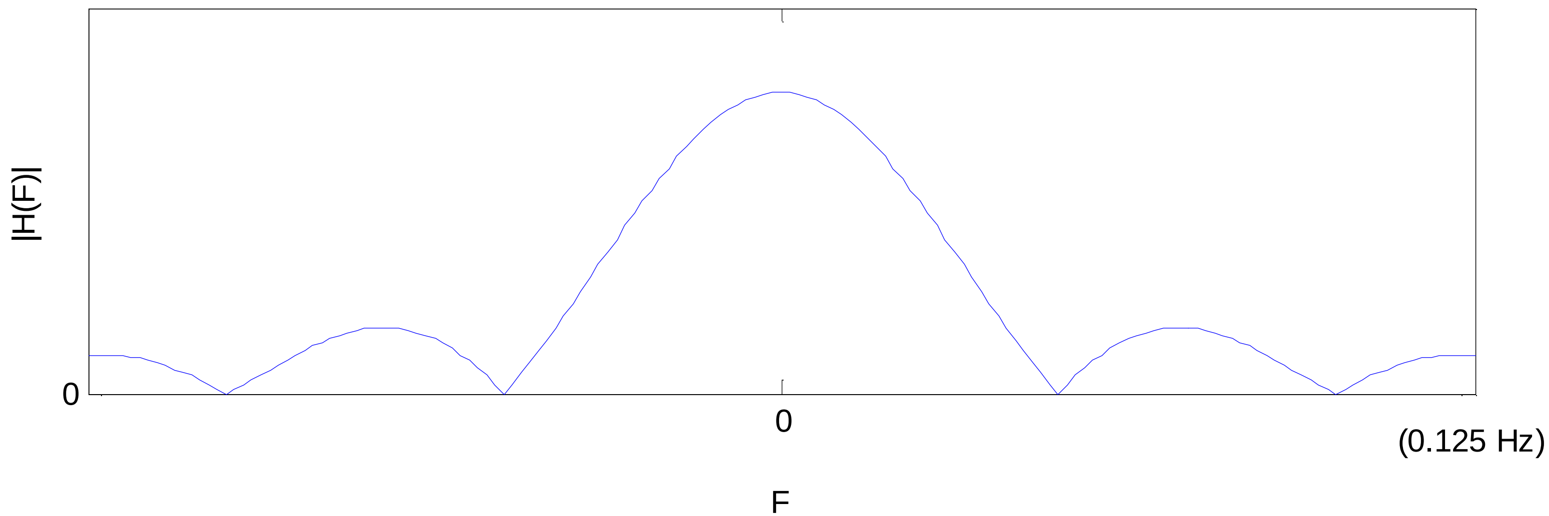


Frequency response for $\mathrm{T} = 320s$



Frequency response for $\mathrm{T} = 640s$

Example 4 (cont.)



Frequency response for $\mathrm{T} = \infty$

- From the previous figures, it is shown that frequency gap becomes smaller when T value is increased. At $T = \infty$, the signal becomes aperiodic signal and the transformation into frequency domain is called Fourier Transform.

Frequency Response for Discrete-Time Signal

- For discrete-time signal, Fourier Series (FS) is called Discrete-Time Fourier Series (DTFS) and Fourier Transform is called Discrete-Time Fourier Transform (DTFT).
- However, in most DSP books and also for this module, both continuous-time and discrete time transformation will be called FS and FT as the objective of the transformation is similar.
- The only difference of the discrete-time signal transformation compare to its continuous-time is coming from the sampling process where $t = nT_s$.

Fourier transformation formulation

Fourier Series Coefficient:

$$a_k = \frac{1}{T}\int_T x(t)e^{-j2\pi kt/T}dt$$

Time Sampling: $t = nT_s$.
Thus, $T = N$ & $F_o = f_o$

DTFS Coefficient:

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-j2\pi kn/N}$$

Replace $\frac{k}{T} = kF_o$ with F

Fourier Transform:

$$H(F) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}dt$$

Time Sampling: $t = nT_s$.
Thus, $Ft = \frac{F}{F_s}n = fn$

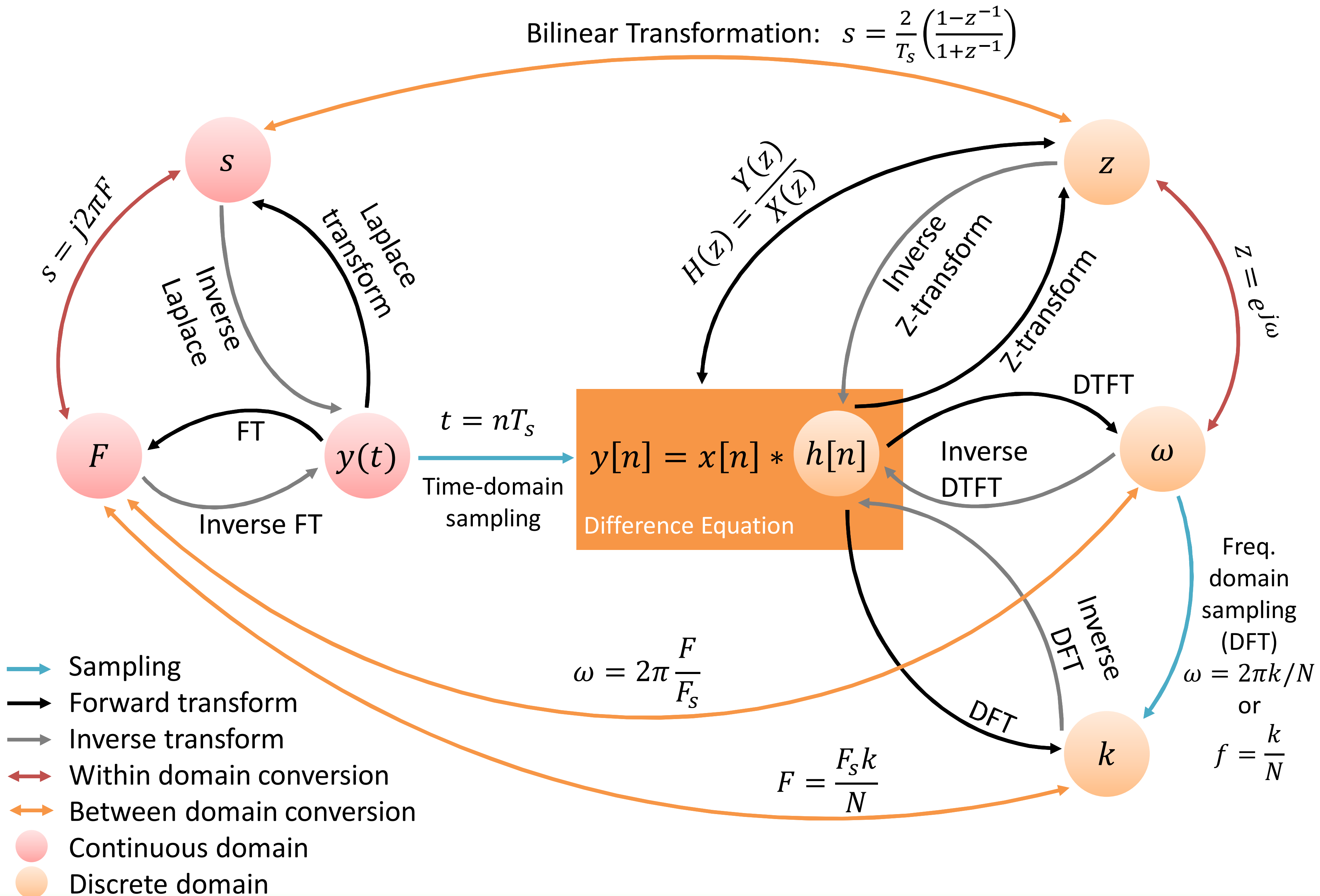
DTFT:

$$H(f) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi fn}$$

Frequency Sampling: $f = k/N$.

DFT:

$$H[k] = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi kn/N}$$


Bilinear Transformation: $s = \frac{2}{T_s}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$
s
z
$s = j2\pi F$
Laplace transform
Inverse Laplace
$H(z) = \frac{Y(z)}{X(z)}$
Inverse Z-transform
Z-transform
$z = e^{j\omega}$
DTFT
F
FT
$y(t)$
Inverse FT
$t = nT_s$
Time-domain sampling
$y[n] = x[n] * h[n]$
Difference Equation
Inverse DTFT
ω
Freq. domain sampling (DFT)
$\omega = 2\pi k/N$
or
$f = \frac{k}{N}$
Inverse DFT
DFT
k
$\omega = 2\pi \frac{F}{F_s}$
$F = \frac{F_s k}{N}$
Sampling
Forward transform
Inverse transform
Within domain conversion
Between domain conversion
Continuous domain
Discrete domain

References

1) John G. Proakis, Dimitris K Manolakis, “Digital Signal Processing: Principle, Algorithm and Applications”, Prentice-Hall, 4th edition (2006).

2) Sanjit K. Mitra, “Digital Signal Processing-A Computer Based Approach”, McGraw-Hill Companies, 3rd edition (2005).

3) Alan V. Oppenheim, Ronald W. Schafer, “Discrete-Time Signal Processing”, Prentice-Hall, 3rd edition (2009).