SEL4223 Digital Signal Processing

Z - Transform

Musa Mohd Mokji

The use of z-transform

- To obtain the difference equation of the LTI system (For FIR system, diff. equation can be obtained from convolution formulation while for IIR system, z-transform is a better technique).

- To evaluate the stability and causality of LTI system (this can be done by analyzing the roots of the polynomial)

Why difference equation is so important?

Impulse Response:

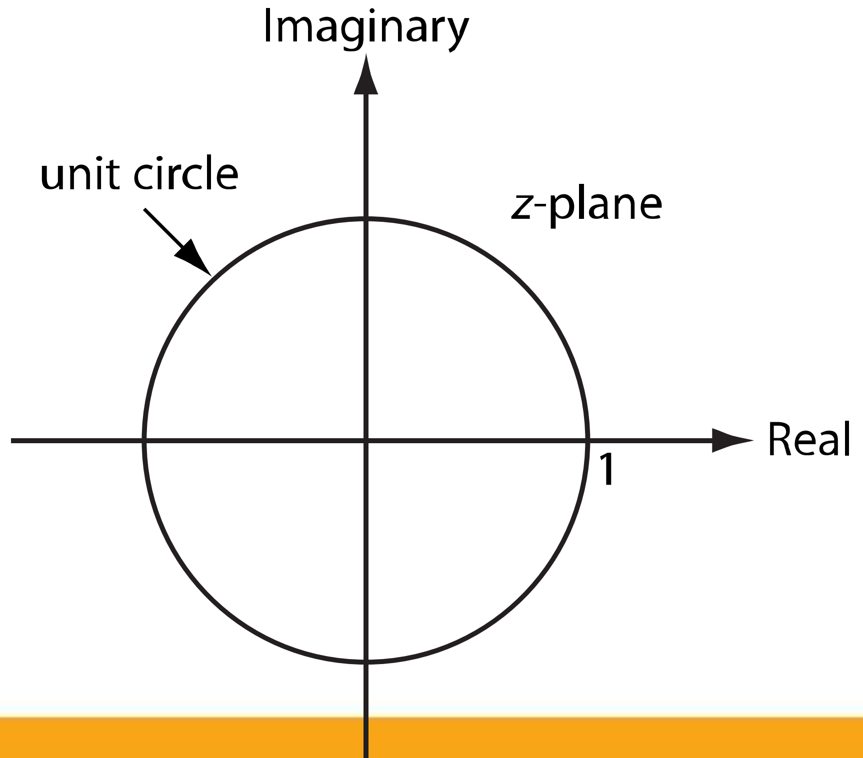
Designing filter means designing the impulse response

Difference Equation:

Needed for implementation (on hardware) as it describe the system in terms of input and output sequence

What is Z-Transform

- Transform a discrete-time signal (in time domain) to z-plane (polynomial).
- z-plane is a complex-plane as shown below;



Z-Transform Formulation

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} \Rightarrow forward\ transform$$

$$h[n] = \frac{1}{2\pi j} \oint_{c} H(z) z^{n-1} dz \Rightarrow inverse\ transform$$

ROC (Region of Convergence)

- A region on the z-plane where the magnitude of $H(z)$ is not ∞ where

$$|H(z)| \leq \alpha \neq \infty$$

- Converge means $H(z)$ has a true value which is not infinity. In other words, $H(z)$ exists.

➢ **Example 1**

$$h[n] = \alpha^n u[n]$$

Solution:

$$H(z) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

Example 1 (cont.)

- $H(z)$ can be rewritten using equation below

$$\sum_{n=N}^{M} A^n = \frac{A^N - A^{M+1}}{1 - A}$$

- Thus,

$$H(z) = \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \frac{(\alpha z^{-1})^0 - (\alpha z^{-1})^{\infty+1}}{1 - (\alpha z^{-1})}$$

- To make sure $H(z) \neq \infty$, $|\alpha z^{-1}|$ must be less than 1 so that $(\alpha z^{-1})^{\infty+1}$ will approximate zero. If $|\alpha z^{-1}| > 1$, $(\alpha z^{-1})^{\infty+1} \approx \infty$ and if $|\alpha z^{-1}| = 1$, the numerator will become zero resulting $H(z) = \infty$.

Example 1 (cont.)

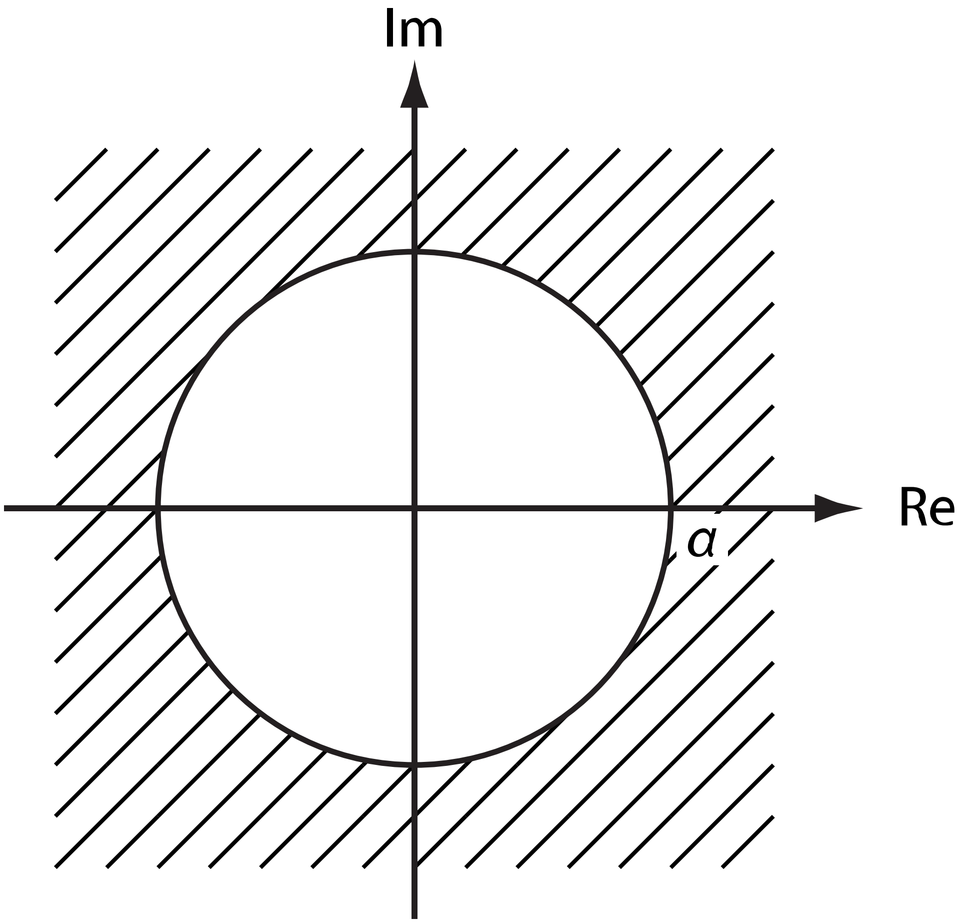
- Thus, $H(z)$ can be written as

$$H(z) = \begin{cases} \dfrac{1}{1 - \alpha z^{-1}} & for\ |z| > \alpha \\ \infty & for\ |z| \leq \alpha \end{cases}$$

- The range where $H(z) \neq \infty$ is called Region of Convergence (ROC).

- Because z is a complex value, it can also be written in polar form where $z = re^{j\theta}$ (r-radius, θ-angle between the real and imaginary value) and $|z| = r$. Thus, the range of the ROC is actually determined by the radius r value.

Example 1 (cont.)



Impulse Response Test

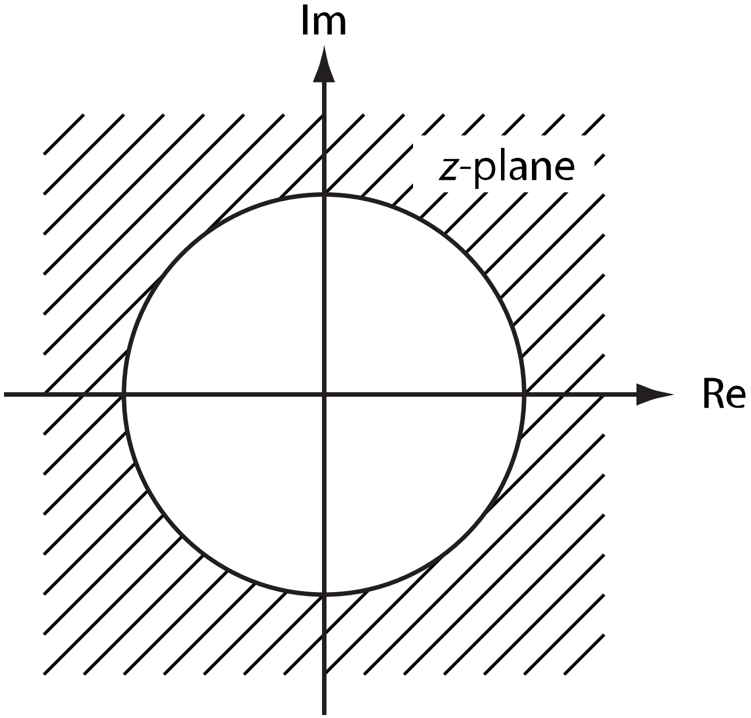
- In previous chapter, it has been shown how the LTI system is evaluated on its stability and causality based on $h[n]$ (impulse response) where

Causal System	Stable System
$h[n] = 0$ for $n < 0$	$\sum_{n=-\infty}^{\infty} \lvert h[n] \rvert \leq \beta \neq \infty$

- z-transform is just another approach on evaluating the stability and causality of the LTI system

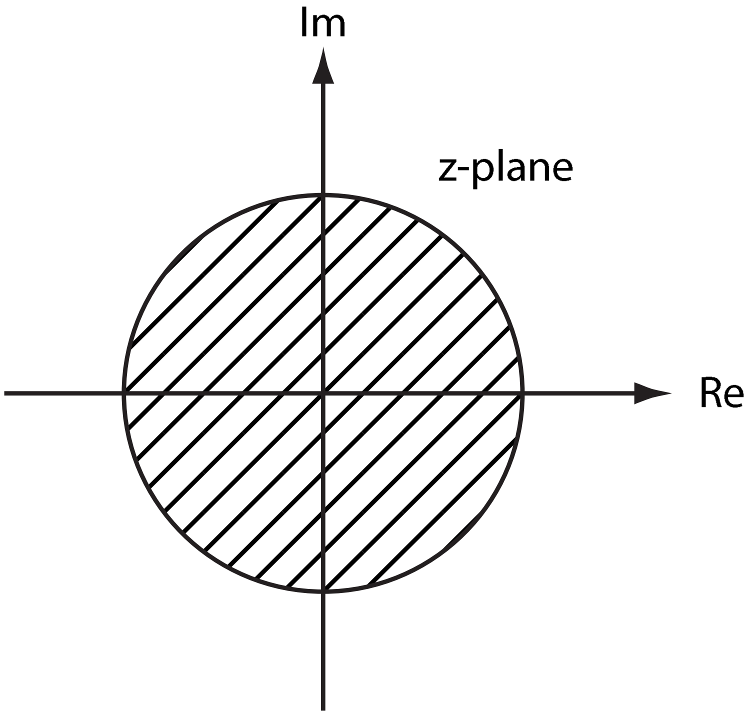
Causality

- Causality of the LTI system will determine the shape of the ROC
- For causal system where $h[n] = 0$ for $n < 0$, ROC is right-handed

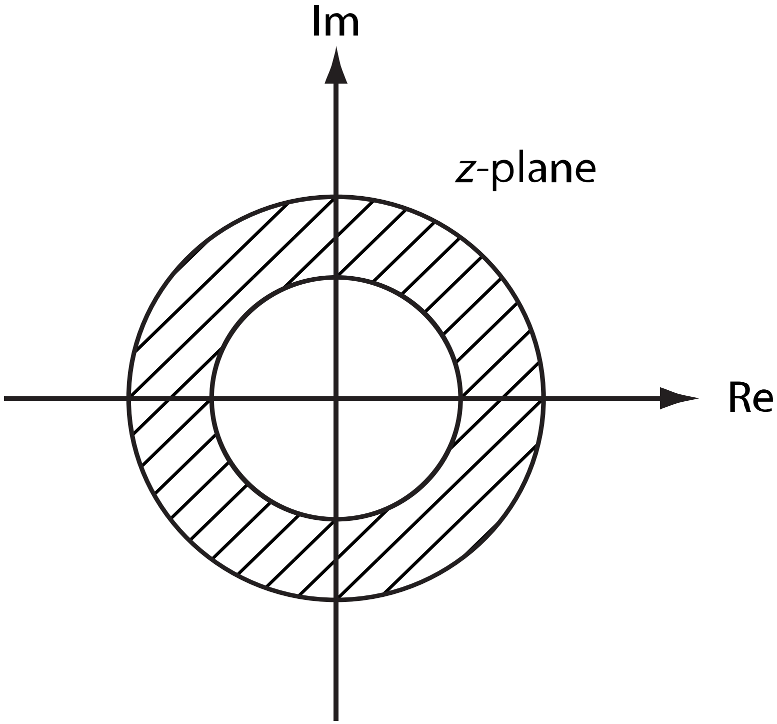


Causality (cont.)

- For anti-causal system where $h[n] = 0$ for $n \geq 0$, ROC is left-handed



- For non-causal system where at $n < 0$ and $n \geq 0$ exist $h[n] \neq 0$, ROC is a ring shaped

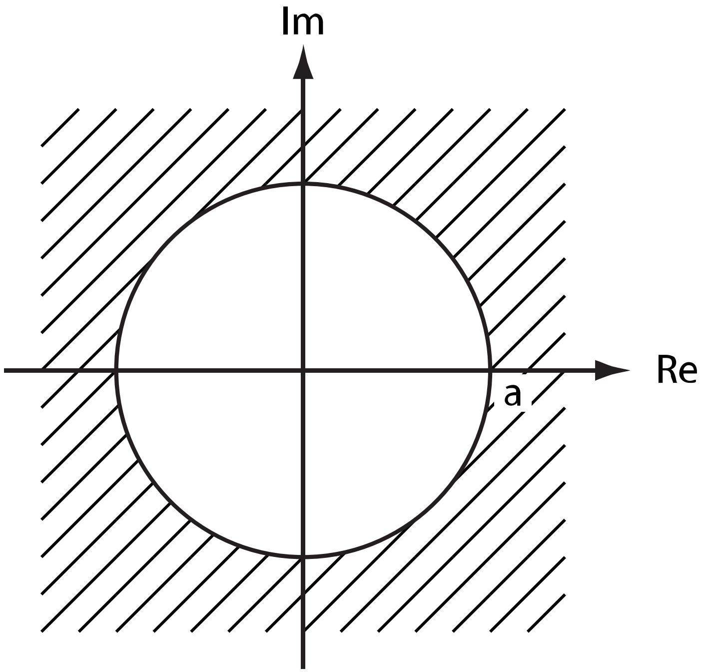


Example 2

- $h[n] = a^n u[n] \Rightarrow$
  this is a causal signal
- $H(z) = \sum_{n=0}^{\infty} a^n z^{-n}$

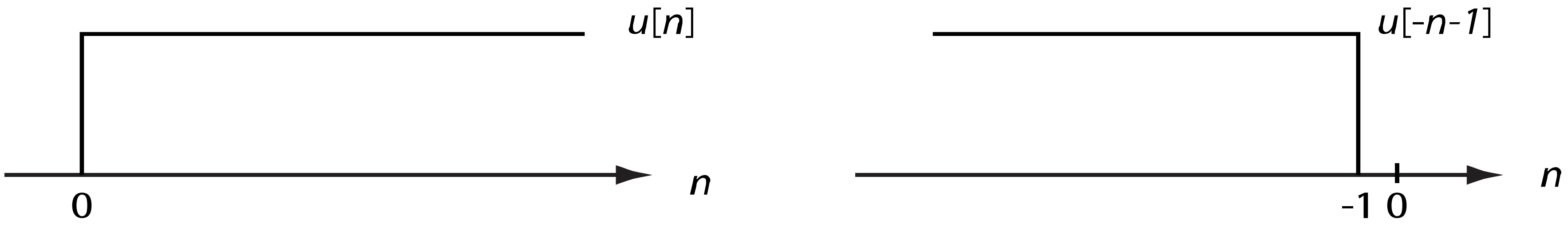
  $= \sum_{n=0}^{\infty} (az^{-1})^n$

  $= \frac{(az^{-1})^0 - (az^{-1})^{\infty+1}}{1-az^{-1}}$
- In order for $H(z)$ to converge, $|az^{-1}| < 1$
- Thus, $H(z) = \frac{1}{1-az^{-1}}$
- ROC: $|z| > |a|$, right-sided ROC



Example 3

- $h[n] = -a^n u[-n-1] \quad \Rightarrow \quad anticausal\ signal$
- Note that the $u[n]$ and $u[-n-1]$ determine the causality of $h[n]$



- $H(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n} \; = \; -\sum_{n=-\infty}^{-1} (az^{-1})^n$

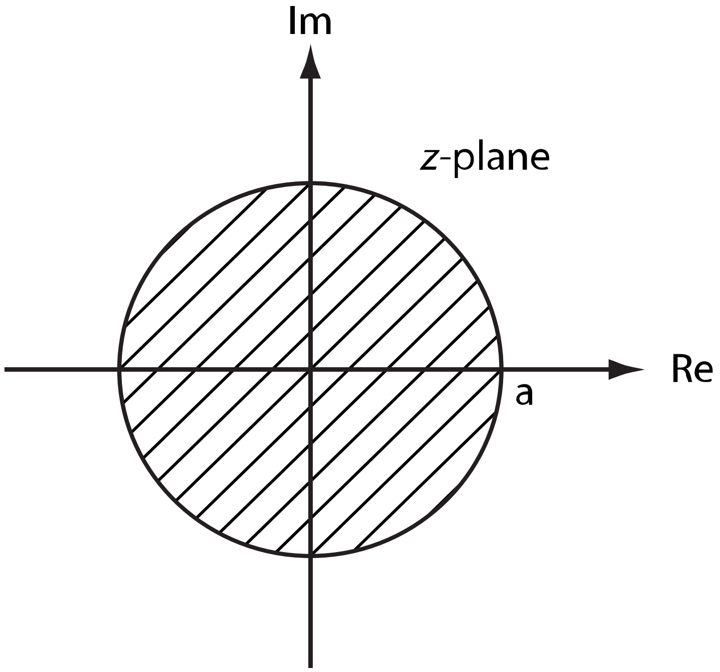
$$= \frac{-(az^{-1})^{-\infty} + (az^{-1})^0}{1 - az^{-1}}$$

Example 3 (cont.)

- To converge, $|az^{-1}| > 1$ so that $(az^{-1})^{-\infty}$ approximately 0. If $|az^{-1}| < 1$, $(az^{-1})^{-\infty}$ will become ∞ and $H(z)$ does not converge.
- Thus, $H(z) = \frac{0+1}{1-az^{-1}}$

$$= \frac{1}{1-az^{-1}}$$

- ROC: $|z| < |a|$
- Note that $H(z)$ in Ex.2 and Ex.3 are similar. The only thing that differentiates the causality of both $H(z)$ is their ROC.



Example 4

- $h[n] = 0.5^n u[n] + 2^n u[-n-1]$

- $H(z) = \sum_{n=-\infty}^{\infty}(0.5^n u[n] + 2^n u[-n-1])z^{-n}$

$$= \sum_{n=0}^{\infty} 0.5^n z^{-n} + \sum_{n=-\infty}^{-1} 2^n z^{-n}$$

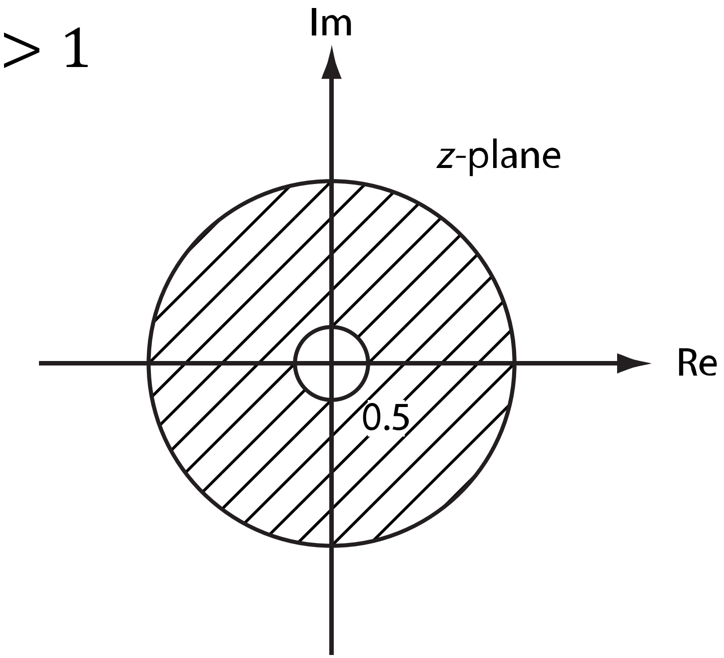
$$= \sum_{n=0}^{\infty}(0.5z^{-1})^n + \sum_{n=-\infty}^{-1}(2z^{-1})^n$$

$$= \frac{(0.5z^{-1})^0 - (0.5z^{-1})^{\infty+1}}{1-0.5z^{-1}} + \frac{(2z^{-1})^{-\infty} - (2z^{-1})^0}{1-2z^{-1}}$$

Example 4 (cont.)

- To converge, $|0.5z^{-1}| < 1$ and $|2z^{-1}| > 1$
- Thus, $H(z) = \frac{1}{1-0.5z^{-1}} - \frac{1}{1-2z^{-1}}$
- ROC:

  $|z| > 0.5$ and $|z| < 2$ or $0.5 < |z| < 2$



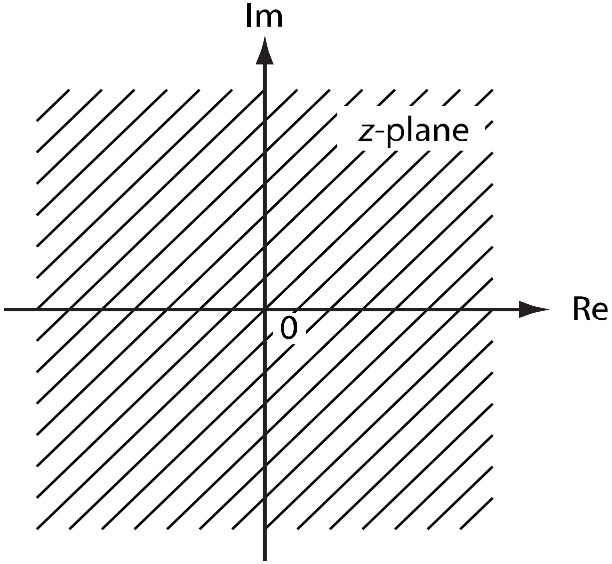
- The three examples are based on an IIR signal of $h[n]$ where either $|M|$ or $|N|$ in equation 3.3 are equal to ∞. Next example is for an FIR system of $h[n]$

Example 5

- $h[n] = \underset{\uparrow}{[1\ 1\ 1\ 1]} \quad \Rightarrow causal$

- $$H(z) = \sum_{n=0}^{3} h[n] z^{-n}$$
  $$= \sum_{n=0}^{3} z^{-n}$$
  $$= 1 + z^{-1} + z^{-2} + z^{-3}$$
  $$= z^{-3}(z^3 + z^2 + z + 1)$$
  $$= \frac{z^3+z^2+z+1}{z^3}$$

- To converge, z can takes any value except $z = 0$. Thus, the ROC is right-sided where $|z| > 0$. In other words the ROC is the entire z-plane except at $z = 0$.



Example 6

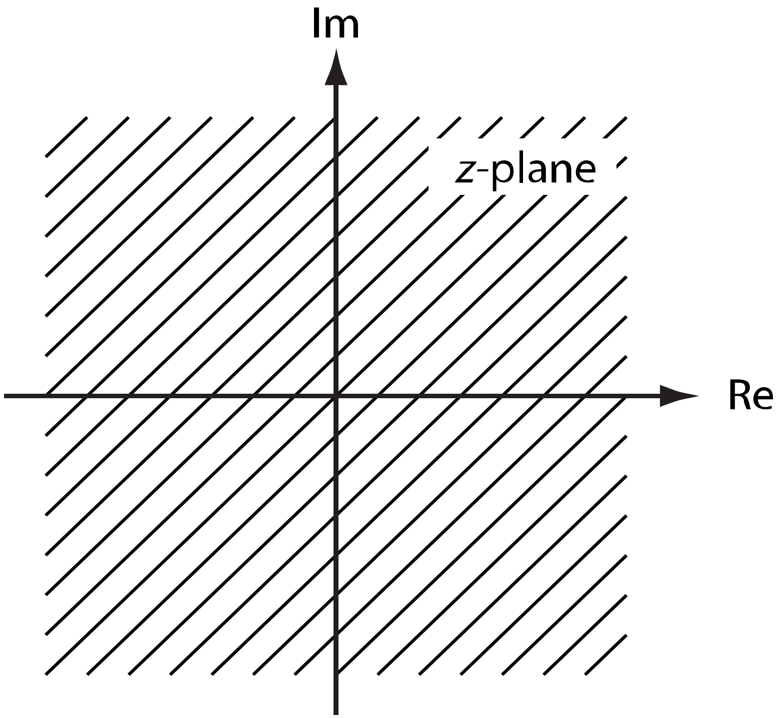
- $h[n] = [1\ 1\ 1\ 1\ 0]$
  $\uparrow$

  $anti - causal$

- $H(z) = \sum_{n=-4}^{-1} z^{-n}$

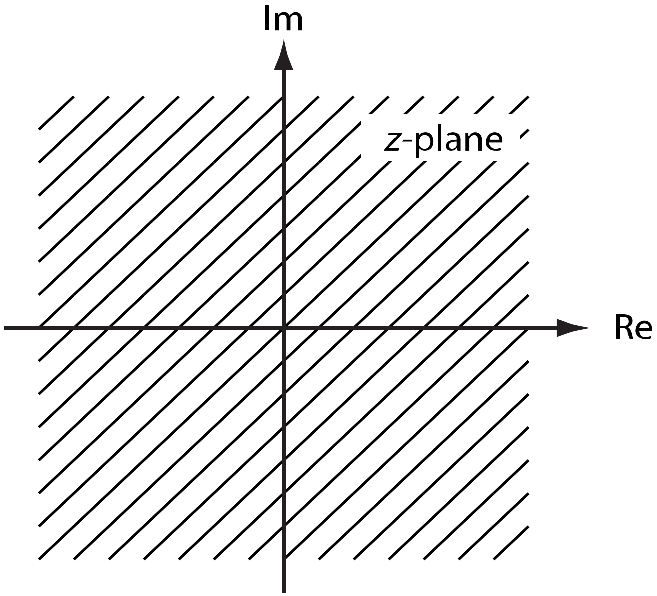
  $= z^4 + z^3 + z^2 + z^1$

- To converge, z can take any value except at $z = \infty$. Thus, the ROC is left-sided $|z| < \infty$. In other words, ROC is the entire z-plane except at $z = \infty$



Example 7

- $h[n] = [1\ 1\ 1\ 1\ 1\ 1] \quad \Rightarrow non - causal\ signal$
  $\uparrow$



- $H(z) = \sum_{n=-3}^{2} z^{-n}$

  $= \sum_{n=-3}^{-1} z^{-n} + \sum_{n=0}^{2} z^{-n}$

  $= z^3 + z^2 + z^1 + \frac{z^2+z+1}{z^2}$

- To converge, z can takes any value except at $z = 0$ and $z = \infty$. The ROC is a ring shape where $0 < |z| < \infty$.

Causality (cont.)

- Conclusion:

  ROC of an FIR system is the entire z-plane except at either $z = 0$ or $z = \infty$ or both depends on the causality of the signal.

Quiz 1

1. $h[n] = [\underset{\uparrow}{1}\ 0\ 1\ 2\ 3]$
2. $h[n] = [0\ 0\ \underset{\uparrow}{1}\ 1\ 0]$
3. $h[n] = [2\ 3\ 1\ \underset{\uparrow}{4}]$
4. $h[n] = \delta[n]$
5. $h[n] = u[n]$
6. $h[n] = u[n + 5]$
7. $h[n] = u[n - 5]$
8. $h[n] = \sum_{k=0}^{\infty} \delta[n - k]$
9. $h[n] = \alpha^n u[n]$
10. $h[n] = \alpha^n u[-n]$
11. $h[n] = \alpha^{n-5} u[n]$
12. $h[n] = \alpha^{n-5} u[n - 5]$
13. $h[n] = 2^n u[n] + 0.8^n u[-n]$

Quiz 1 (cont.)

14. $h[n] = 2^n u[-n] + 0.8^n u[n]$

15. $h[n] = \delta[n] + \delta[n - \infty]$

16. $h[n] = [\underset{\uparrow}{1}\ \infty]$

17. $h[n] = \alpha^2 u[n]$

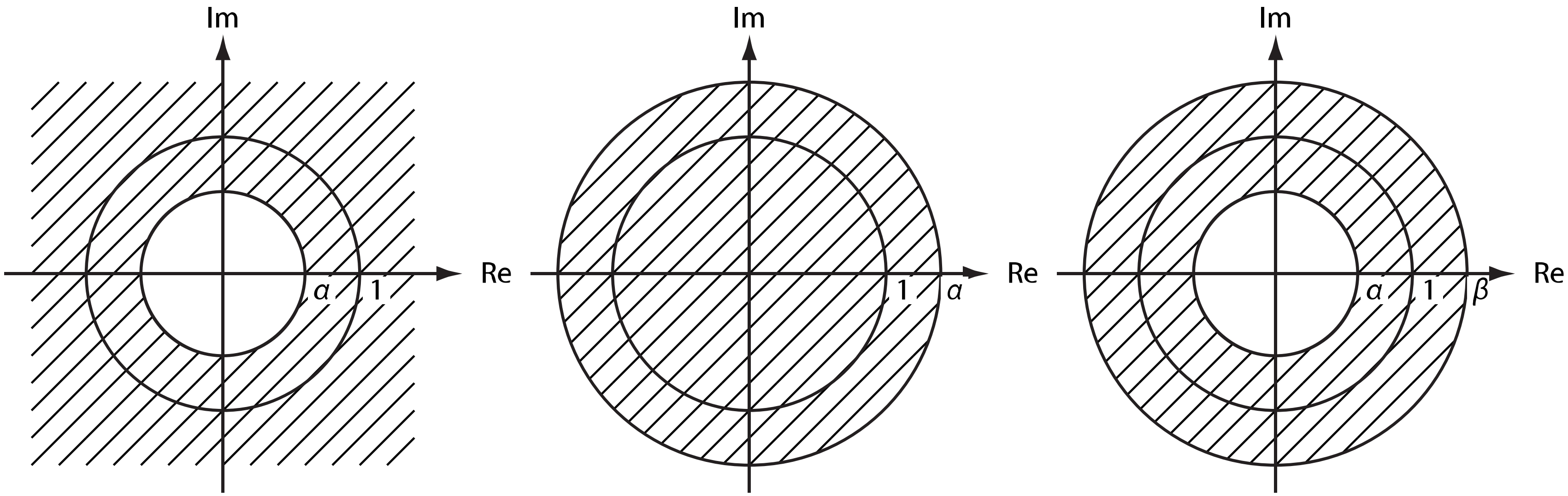
18. $h[n] = \alpha^n u[n] + \alpha^n u[-n - 1]$

19. $h[n] = \alpha^n u[n] + \alpha^{-n} u[-n - 1]$

20. $h[n] = e^{j\pi n} u[n]$

Stability

- For a stable system, unit circle must be inside the ROC.



Exterior ROC
$\alpha < 1$

Interior ROC
$\alpha > 1$

Ring ROC
$\alpha < 1 \,\&\, \beta > 1$

- For FIR system, ROC is the entire z-plane except at 0 and ∞. Thus, unit circle is always inside the ROC. This means that, FIR system is always stable.

Poles and Zeros

- Zeros : z values that cause $H(z) = 0$

  Or z values that result the numerator of $H(z)$ equals to 0.

- Poles: z values that cause $H(z) = \infty$

  Or z values that result the denominator of $H(z)$ equals to 0.

- $H(z) = \frac{1+z^{-1}+z^{-2}+z^{-3}+\cdots+z^{-M}}{1+z^{-1}+z^{-2}+z^{-3}+\cdots+z^{-N}} = \frac{numerator}{denumerator}$

- In general, $H(z)$ can be written as

- $H(z) = \frac{\prod_{m=1}^{M}(z-\beta_m)}{\prod_{n=1}^{N}(z-\alpha_n)}$ Where $\beta_m = zeros$ and $\alpha_n = poles$

Example 8

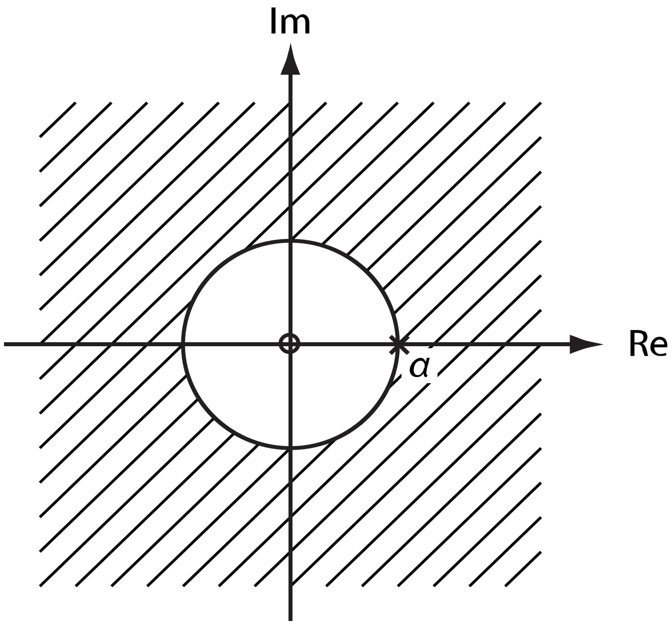
- $h[n] = \alpha^n u[n]$
- $H(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n}$

$$= \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \frac{1}{1-\alpha z^{-1}}$$

- Finding poles & zeroes are easier when $H(z)$ is written in z instead of z^{-1}, thus,
- $H(z) = \frac{1}{z^{-1}(z-\alpha)}$

$$= \frac{z}{z-\alpha}$$

- Zeros, $z = 0$
- Poles, $z - \alpha = 0, z = \alpha$
- ROC, $|z| > \alpha$



O – Zeros, X – Poles

Example 9

- $h[n] \;= \delta[n] - 3\delta[n-1] + 2\delta[n-2]$
- $H(z) = \sum_{n=-\infty}^{\infty}(\delta[n] - 3\delta[n-1] + 2\delta[n-2])z^{-n}$

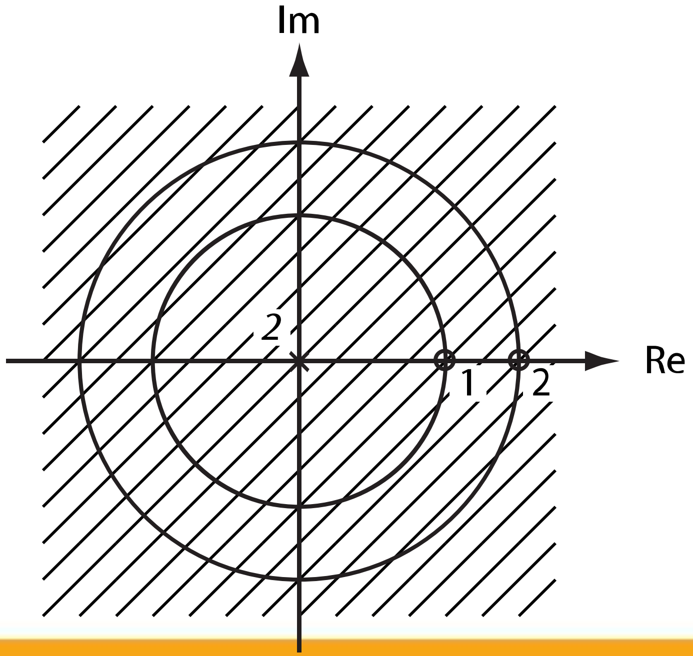
$= \sum_{n=-\infty}^{\infty} \delta[n]z^{-n} - \sum_{n=-\infty}^{\infty} 3\delta[n-1]z^{-n} + \sum_{n=-\infty}^{\infty} 2\delta[n-2]z^{-n}$

$= z^0 - 3z^{-1} + 2z^{-2}$

$= z^{-2}(z^2 - 3z + 2)$

$= \frac{z^2-3z+2}{z^2}$

$= \frac{(z-1)(z-2)}{z^2}$



Characteristic of ROC

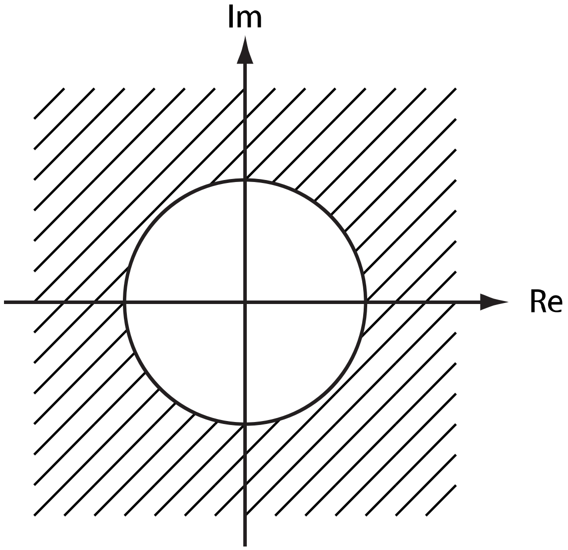
- Boundary of ROC is a circle centered at $z = 0$

- Only poles will determine the ROC where zeros gives no effect to the ROC

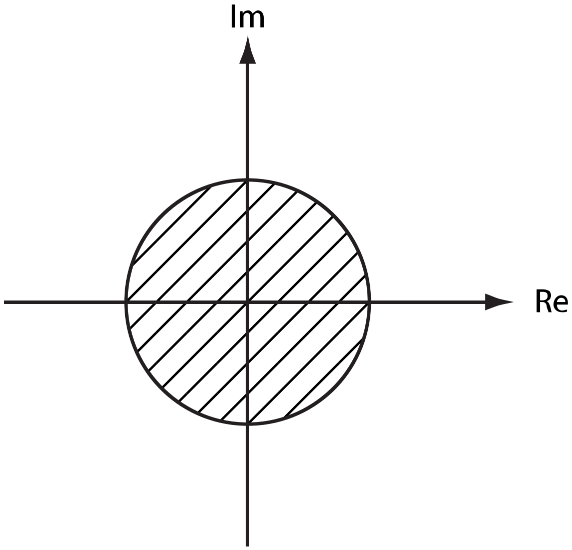
- ROC is a region between poles. Thus, there must be no poles inside ROC

- ROC is a connected region. Thus, only three shapes of ROC are valid, or else there will be no ROC for the LTI system

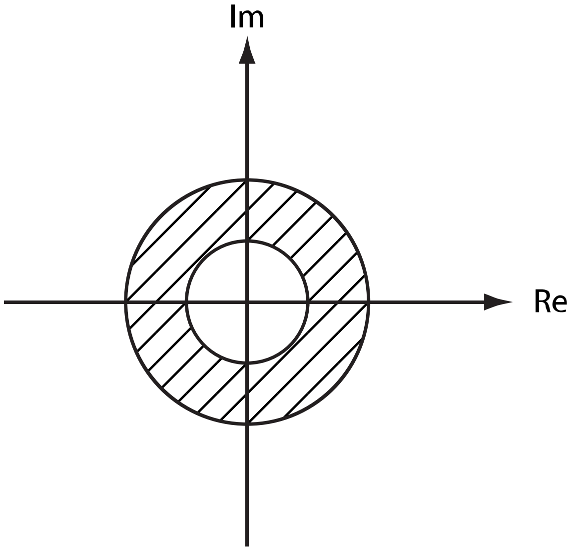
Valid ROC Shape



Outer ROC



Inner ROC



Ring ROC

Characteristic of ROC (cont.)

- Outer ROC is for right-sided signal (causal signal)
- Inner ROC is for left-sided signal (anti-causal signal)
- Ring ROC is for two-sided signal (combination of causal and anti-causal signal)
- For finite length signal, ROC is the entire z-plane except at $z = 0$ or $z = \infty$ or both
- If ROC is inclusive of the unit circle, the system is stable and Fourier Transform for the LTI system exists.

References

1) John G. Proakis, Dimitris K Manolakis, "Digital Signal Processing: Principle, Algorithm and Applications", Prentice-Hall, 4th edition (2006).

2) Sanjit K. Mitra, "Digital Signal Processing-A Computer Based Approach", McGraw-Hill Companies, 3rd edition (2005).

3) Alan V. Oppenheim, Ronald W. Schafer, "Discrete-Time Signal Processing", Prentice-Hall, 3rd edition (2009).