SEL4223 Digital Signal Processing

Discrete-Time System

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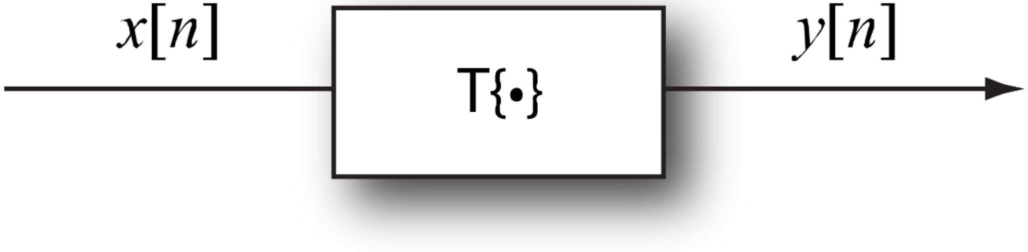
What is a System

- An entity that changes or transforms a signal (input signal) into another form of signal (output signal) based on specific transfer function.

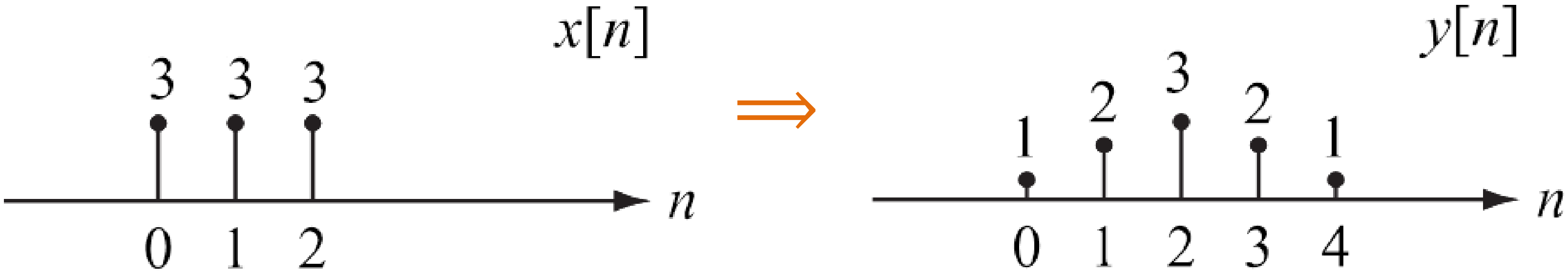
$$y[n] = T\{x[n]\}$$



Example 1: Moving Average System

- Average of 3 samples

$$y[n] = \frac{1}{3}\{x[n] + x[n-1] + x[n-2]\}$$

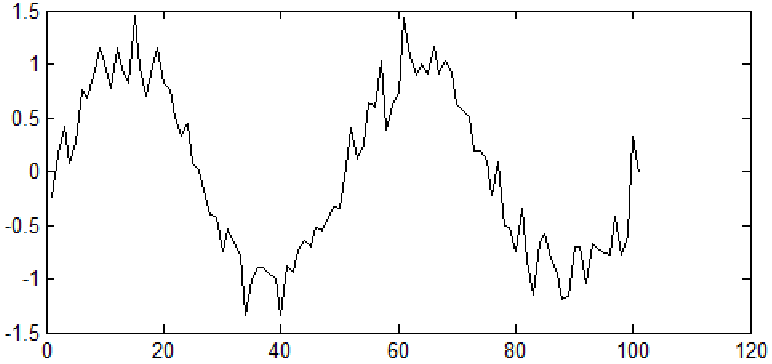


- This system changes the consistent value of input signal $x[n]$ into a new gradual value of output signal $y[n]$. This process is commonly known as smoothing.

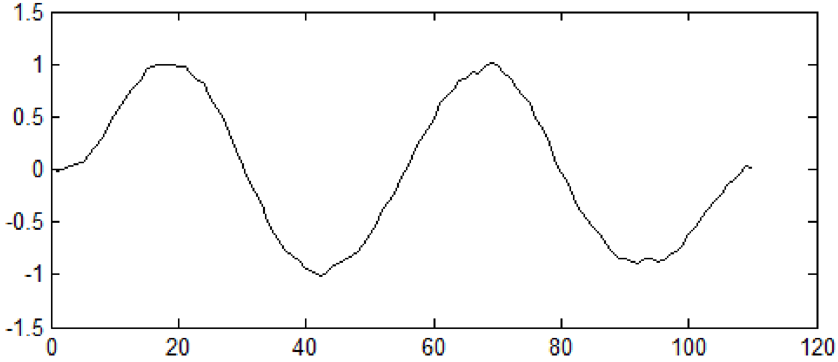
Example 1: Moving Average System

- Average of 10 samples

$$y[n] = \frac{1}{10}\sum_{k=0}^{9} x[n-k]$$



$x[n]$



$y[n]$

Characteristics of a Discrete-Time system

1. Linearity
2. Time-invariant or time-variant
3. Causality
4. Stability
5. Memory

- Linearity and Time-invariant are to ensure consistency at the output of a system
- Causality and Stability are to ensure the practicality of implementing the system

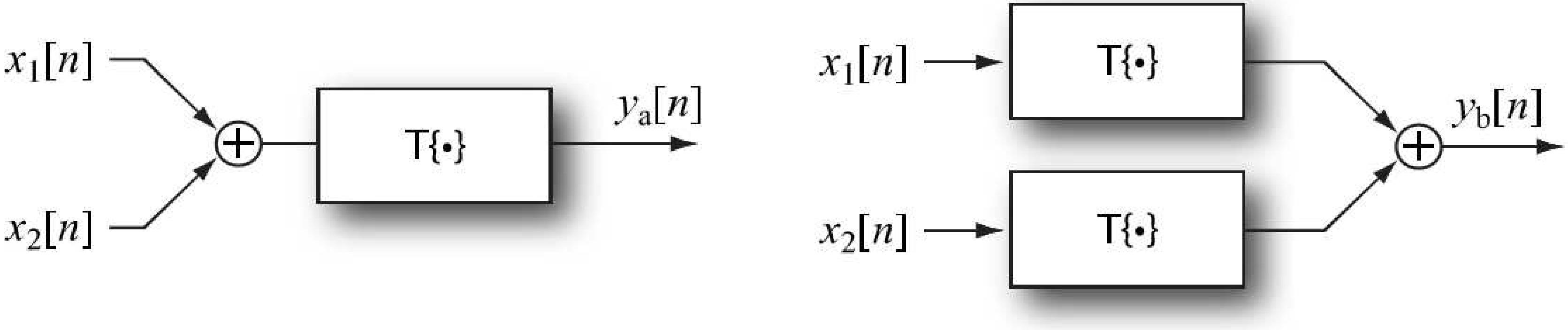
Linearity (for consistent output)

- Obeys the superposition principle.

$$y_a[n] = T\{x_1[n] + x_2[n]\}$$

$$y_b[n] = T\{x_1[n]\} + T\{x_2[n]\}$$

$$y_a[n] = y_b[n]$$



Example 2

- $y[n] = x[n]^2$

- $y_a[n] = (x_1[n] + x_2[n])^2$

$$= x_1\ [n]^2 + 2x_1[n]x_2[n] + x_2\ [n]^2$$

- $y_b[n] = x_1[n]^2 + x_2[n]^2$

- $y_a[n] \neq y_b[n]$

\# The system is not linear

Example 3

- $y[n] = x[n] + 2x[n-1]$
- $y_a[n] = (x_1[n] + x_2[n]) + 2(x_1[n-1] + x_2[n-1])$

  $= x_1[n] + x_2[n] + 2x_1[n-1] + 2x_2[n-1]$
- $y_b[n] = x_1[n] + 2x_1[n-1] + x_2[n] + 2x_2[n-1]$

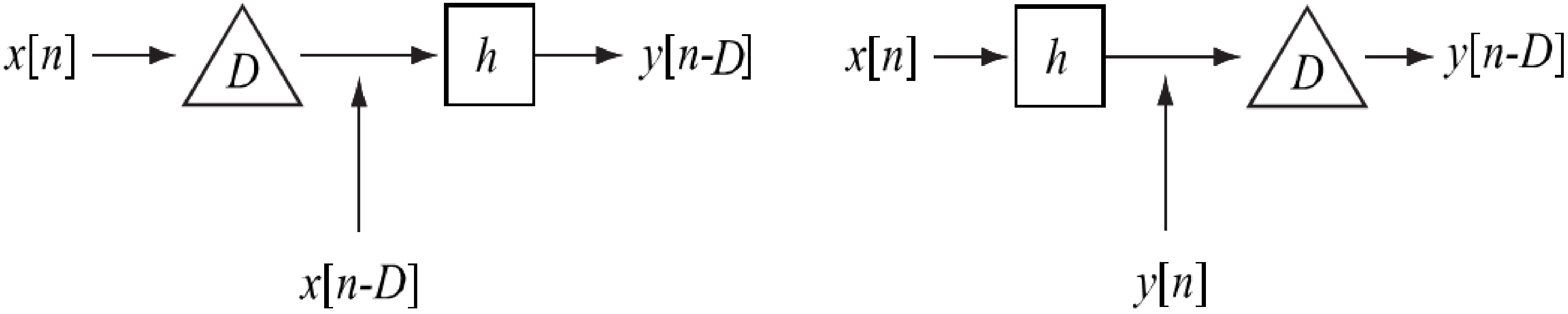
- $y_a[n] = y_b[n]$

\# The system is linear

Time-Invariant (for consistent output)

- Output to the system is similar when a specific delay is applied either to the input or output of the system.

$$T\{x[n-D]\} = y[n-D]$$



Example 4

- $y[n] = \frac{1}{2}(x[n] + x[n-1])$
- $T\{x[n-n_d]\} = \frac{1}{2}(x[n-n_d] + x[n--1-n_d])$
- $y[n-n_d] = \frac{1}{2}(x[n-n_d] + x[n-n_d-1])$
- Result when delayed at the inputs is similar when the delayed at the outputs.

\# $y[n]$ is time-invariant

Tips

To test $T\{x[n-n_d]\}$, minus whatever inside the input signal bracket with n_d.

To test $y[n-n_d]$, change all n with $n-n_d$.

Example 5

- $y[n] = \sum_{k=0}^{n} x[k]$

- $T\{x[n-n_d]\} = \sum_{k=0}^{n} x[k-n_d]$
- Let $m = k - n_d$
- $T\{x[n-n_d]\} = \sum_{m=-n_d}^{n-n_d} x[m] \; = \sum_{k=-n_d}^{n-n_d} x[k] \; \rightarrow (a)$
- $y[n-n_d] = \sum_{k=0}^{n-n_d} x[k] \rightarrow (b)$

- $(a) \neq (b)$, thus the system is not time-invariant.

Example 5 (cont.)

- To prove the answer, let's repeat the example by giving values for the input signal as $x[n] = [1, \underset{\uparrow}{2}, 1]$ and the delay as $n_d = 1$:

- $x[n - n_d] = x[n-1] = [\underset{\uparrow}{1}, 2, 1]$

- $T\{x[n-1]\} = \sum_{k=0}^{n} x[n-1] = [\underset{\uparrow}{1}, 1+2, 1+2+1, \dots, 1+2+1]$

$$= [\underset{\uparrow}{1}, 3, 4, \dots, 4]$$

- $y[n] = \sum_{k=0}^{n} x[k] = [1+2, \underset{\uparrow}{2}, 2+1, \dots, 2+1] = [3, \underset{\uparrow}{2}, 3, \dots, 3]$

Example 5 (cont.)

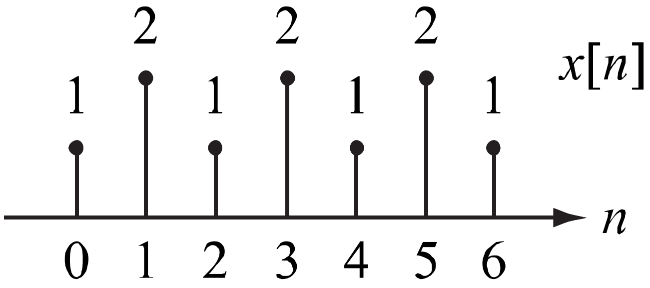
- $y[n-n_d] = y[n-1] = [\underset{\uparrow}{3}, 2, 3, \dots, 3]$

- The system is proved to be not time-invariant as $T\{x[n-1]\} \neq y[n-1]$.

Example 6

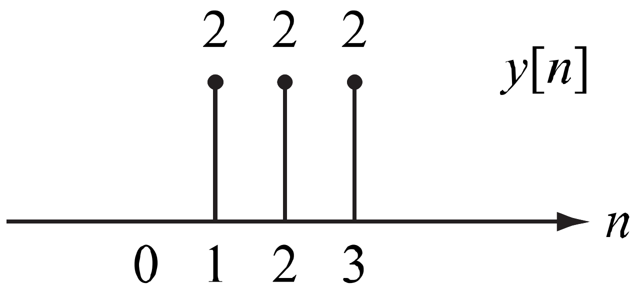
- $y[n] = x[2n]$
- $T\{x[n - n_d]\} = x[2n - n_d] \rightarrow (a)$
- $y[n - n_d] = x[2(n - n_d)] = x[2n - 2n_d] \rightarrow (b)$
- $(a) \neq (b)$, thus the system is not time-invariant.

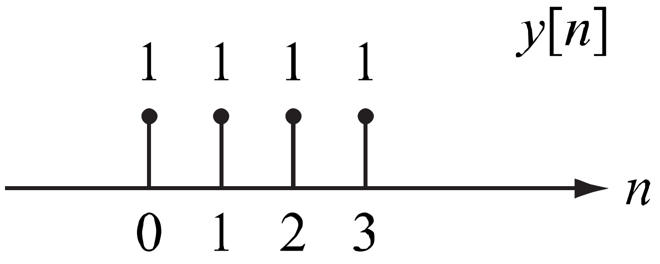
- For this example, let's say that $x[n]$ is as below



Example 6 (cont.)

- $T\{x[n-n_d]\}$ test: First delay the input $x[n]$ by 1 sample and do the operation, the result is
- $y[n-n_d]$ test: Do the operation first and then delay the output $y[n]$ by 1 sample, the result is





- Results are not similar, thus it is time-variant as the output changes with the delay.

Time-invariant (cont.)

- From the last 2 examples, the time-invariant test is actually testing the system with input signal at two different delay values.
  - First test is with delay $= n_d$
  - Second test is with delay $= 0$
- Thus, if n_d is to be replaced with real value, it must be $n_d \neq 0$ as this will gives both test using delay $= 0$.
- In conclusion, time-invariant system can also be described as a system that produces consistent output even if the input signal is feed to the system at different delay (time).

Causality (for practical implementation)

- Causal system means the system only involves current and past input values.

Current value is at time n

Past values are at time $n - k$

Future values are at time $n + k$

- Causality is important when dealing with online system because the system does not know what the future value is. Thus, it is impossible to compute the unknown value. However, non-causal system is not a problem to an offline system where input signal has been stored earlier.

Example 7

1. $y[n] = \frac{1}{3}(x[n] + 2x[n-1] + 2x[n-2])$

   This is a causal system, since no future values are involved.

2. $y[n] = (\alpha)^{n-1}u[n]$

   Time is only represented by the square brackets[ ]. Thus, only current values are involved. This system is causal.

3. $y[n] = \sum_{k=-2}^{1} x[n-k]$

   If we expand above equation,
   $y[n] = x[n+2] + x[n+1] + x[n] + x[n-1]$.
   The first 2 x are future values. Thus, the system is not causal.

Stability (for practical implementation)

- A system is said to be stable if a bounded input to the system will produce bounded output – BIBO.

$$|y[n]| \ and \ |x[n]| < \infty$$

Where $x[n]$ and $y[n]$ does not have a value of ∞ at all values of n.

- Stability is important for practicality because no computer or any digital processor can compute using ∞ value. When computation cannot be done, the system will hang or crash. So, it is important to keep the system stable by avoiding the ∞ value.

Example 8

1. $y[n] = \frac{1}{3}(2x[n] + x[n-1] + 2x[n-1])$

   Output $y[n]$ only depends on 3 values of $x[n]$ at different n. If $x[n]$ is bounded, output will also be bounded. Thus the system is stable.

2. $y[n] = \sum_{k=0}^{n} x[k]$

   In this example, bounded input does not ensure bounded output.

   – If $x[n] = u[n]$, a unit step and when $n = \infty$, $y[\infty]$ will have ∞ value, where $y[\infty] = 1 + 1 + 1 + \cdots + 1 = \infty$. The system is not stable.

   – If $x[n] = \delta[n]$, an impulse signal, $y[n]$ will always be equal to 1 for any value of n. Thus, the system is stable.

Memory

- A system is said to have a memory if it is consist of non-current input.
- A system that consist only current input is called memory-less system.

Quiz 1

- Is the signal bounded or not?

1. $x[n] = u[n]$
2. $x[n] = \delta[n]$
3. $x[n] = k\delta[n-k]$
4. $x[n] = \sum_{k=0}^{n} \delta[k]$
5. $x[n] = 2^n$
6. $x[n] = 0.5^n$
7. $x[n] = 0.1^n u[n]$
8. $x[n] = \left(\frac{2}{3}\right)^n u[-n]$
9. $x[n] = 10^{2n} u[-n-1]$
10. $x[n] = 0.2^{n-10} u[n]$
11. $x[n] = nu[n]$

Quiz 2

Determine the linearity, causality, stability and time-invariance of these systems;

1. $y[n] = x[n^2]$
2. $y[n] = x[n] + nx[n]$
3. $y[n] = (2n - 3)x[n]$
4. $y[n] = px[n] + x[n - p]$
5. $y[n] = \sum_{k=0}^{2} x[n - k]$
6. $y[n] = \sum_{k=-1}^{1} x[kn]$
7. $y[n] = \sum_{k=-2}^{0} x[kn - k]$
8. $y[n] = (x[n])^2$
9. $y[n] = 2^{n-2}x[n]$
10. $y[n] = \sum_{k=0}^{2} x[k]\,\delta[n - k]$
11. $y[n] = \alpha x[n] + \beta x[n - 2]$
12. $y[n] = \sum_{k=0}^{M} \alpha_k x[n - k]$
13. $y[n] = \sum_{k=0}^{n} \alpha_k x[n - k]$

References

1) John G. Proakis, Dimitris K Manolakis, “Digital Signal Processing: Principle, Algorithm and Applications”, Prentice-Hall, 4th edition (2006).

2) Sanjit K. Mitra, “Digital Signal Processing-A Computer Based Approach”, McGraw-Hill Companies, 3rd edition (2005).

3) Alan V. Oppenheim, Ronald W. Schafer, “Discrete-Time Signal Processing”, Prentice-Hall, 3rd edition (2009).