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Conformal Mapping and its Applications

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Conformal Mapping and its Applications

Outline:

- Conformality
- Bilinear transformation, Symmetry principle
- Schwarz-Christoffel transformation, Riemann map
- Boundary Value Problems, Equipotentials, Streamlines
- Electrostatics, Heat Flow, Fluid Mechanics
- Airfoil, Joukowski transformation

Geometric Meaning of Complex Functions

- The graph of a real-valued function of a real variable can often be displayed on a two-dimensional coordinate diagram.
- However, for $w = f(z)$, where z and w are complex variables, a graphical representation of the function f would require displaying a collection of four real numbers in a four-dimensional coordinate diagram.
- Since this is not accessible to our geometric visualization, some alternatives are called for.

Geometric Meaning of Complex Functions

- A commonly used graphical representation of a complex-valued function of a complex variable, consists in drawing the domain of definition (z-plane) and the domain of values (w-plane) in separate complex planes.
- The function $w = f(z)$ is then regarded as a mapping of points in the z-plane onto points in the w-plane.
- The point w is also called the image of the point z.
- More information is usually exhibited by sketching images of specific families of curves in the z-plane.

Conformal Mapping

- A mapping with the property that angles between curves are preserved in magnitude as well as in direction is called a *conformal mapping*.
- Thus any set of orthogonal curves in the z-plane would therefore appear as another set of orthogonal curves in the w-plane.
- Conformal mapping function can be found in the class of *analytic function* subject to certain conditions.

Theorem

Let the function f be analytic on a region D of the complex plane and let its derivative f' has no zeros there. Then the mapping defined by f is conformal in D.

Conformal Mapping and Laplace's Equation

- The Laplace's equation is invariant under conformal mapping.
- This forms the basis of a method of solving numerous two-dimensional boundary-value problems such as the Dirichlet problem and the Neumann problem.
- In various applied problems, by means of conformal maps, problems for certain "physical regions" are transplanted into problems on some standardized "model regions" where they can be solved easily.
- By transplanting back we obtain the solutions of the original problems in the physical regions.
- This process is used, for example, for solving problems about fluid flow, electrostatics, heat conduction, mechanics, and aerodynamics. These applications of conformal maps will be discussed later.

Conformal Mapping and Dirichlet Problem

Let Ω be a simply connected region in the complex plane with boundary Γ, and let ϕ be a continuous real-valued function on Γ. The Dirichlet problem consists in finding a function u satisfying the conditions:

1. u is continuous in $\Omega \cup \Gamma$.
2. u is harmonic in Ω.
3. $u = \phi$ on Γ.

It can be shown that the function u has the form

$$u(z) = \frac{1}{2\pi i} \int_\Gamma \phi(w) \frac{1 - |f(z)|^2}{|f(w) - f(z)|^2} \frac{f'(w)}{f(w)}\, dw, \quad z \in \Omega,$$

where f is a one-to-one analytic function that maps Ω onto a unit disk. The integral in the formula above is a complex integral.

Some Types of Conformal Mapping

There are various classes of conformal mappings that frequently arise in applications. Some of these are:

- Moebius Transformations
- Schwarz-Christoffel Mapping
- Riemann Map

Moebius Transformation

Definition

A Moebius transfomation (MT) is function defined by

$$w = f(z) = \frac{az + b}{cz + d},$$

where a, b, c, d are complex constants such that $ad \neq bc$.

- For $c \neq 0$, MT has a simple pole at $z = -d/c$.
- $$\frac{dw}{dz} = \frac{ad - bc}{(cz + d)^2} \neq 0$$
- MT is also known as a *fractional linear transformation*.
- Since MT $\implies cwz + dw - az - b = 0$,which is linear in both w and z, MT is also known as a *bilinear transformation*.

The Linear Function

- The linear function

$$f(z) = z + b,$$

  where b is a complex constant, always describes a *translation*.
- The linear function

$$f(z) = az, \quad a \neq 0, 1,$$

  where a is a complex constant, always describes a *rotation* and a *magnification*.
- Thus the linear function

$$w = f(z) = az + b$$

  can be considered as a mapping which comprises of translation, rotation and magnification.

The Inverse Function

The inverse transformation is $w = f(z) = 1/z$.

- The image of a line under the inverse transformation is either a line or a circle.
- The image of a circle under the inverse transformation is either a line or a circle.
- If we think of a straight line as a circle with infinite radius, then the set of circles and straight lines is known as the *generalized circles*.
- The inverse transformation $w = 1/z$ maps generalized circles to generalized circles.

MT and Generalized Circles

- Observe that MT may be written as

$$w = \frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d) + \frac{bc-ad}{c}}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c} \cdot \frac{1}{cz+d}.$$

- This shows that MT is a series of several elementary transformations: rotation, magnification, and inversion.
- Note that a linear transformations maps straight lines to straight lines, and circles to circles, while the inverse transformation maps generalized circles to generalized circles.
- Thus MT must also maps generalized circles to generalized circles.

General Rule

Suppose:

- Γ: Generalized Circle (line or circle)
- BLT: $w = f(z) = \dfrac{az+b}{cz+d}, \qquad ad \neq bc.$
- Therefore f has a simple pole at$z = -\frac{d}{c}$.

General Rule:

- $z = -d/c \in \Gamma \implies f(-d/c) = \infty \implies$ The image of G is unbounded $\implies f(\Gamma)$ is a straight line.
- $z = -d/c \notin \Gamma \implies f(G)$ is bounded $\implies f(\Gamma)$ is a circle.

Note:

- Two points determine a line.
- Three points determine a circle.

Three Points Determine a Circle (Formula)

The center $z_0 = x_0 + iy_0$ of the circle through

$$z_1 = x_1 + iy_1, z_2 = x_2 + iy_2, z_3 = x_3 + iy_3$$

satisfies the simultaneous equation

$$\begin{aligned} 2(x_1 - x_2)x_0 + 2(y_1 - y_2)y_0 &= |z_1|^2 - |z_2|^2, \\ 2(x_1 - x_3)x_0 + 2(y_1 - y_3)y_0 &= |z_1|^2 - |z_3|^2. \end{aligned}$$

The radius is given by $r = |z_0 - z_1| = |z_0 - z_2| = |z_0 - z_3|$.
Therefore the equation of the circle is $|z - z_0| = r$.

Three Points Determine a Circle (Proof)

Since z_1 and z_2 are equidistant to the center z_0, we have

$$\begin{aligned}
|z_1 - z_0| &= |z_2 - z_0| \\
|z_1 - z_0|^2 &= |z_2 - z_0|^2 \\
(z_1 - z_0)\overline{(z_1 - z_0)} &= (z_2 - z_0)\overline{(z_2 - z_0)} \\
(z_1 - z_0)(\overline{z_1} - \overline{z_0}) &= (z_2 - z_0)(\overline{z_2} - \overline{z_0}) \\
|z_1|^2 - z_1\overline{z_0} - \overline{z_1}z_0 + |z_0|^2 &= |z_2|^2 - z_2\overline{z_0} - \overline{z_2}z_0 + |z_0|^2 \\
|z_1|^2 - |z_2|^2 &= (z_1 - z_2)\overline{z_0} + (\overline{z_1} - \overline{z_2})z_0 \\
&= (z_1 - z_2)\overline{z_0} + \overline{(z_1 - z_2)\overline{z_0}} \\
&= 2\mathrm{Re}\,(z_1 - z_2)\overline{z_0} \\
&= 2(x_1 - x_2)x_0 + 2(y_1 - y_2)y_0.
\end{aligned}$$

Repeat the previous calculation with z_3 in place of z_2 gives

$$|z_1|^2 - |z_3|^2 = 2(x_1 - x_3)x_0 + 2(y_1 - y_3)y_0.$$

Finding Specific MT

- Previous Problem: Given MT, determine the image in the w-plane of a given generalized circle in a z-plane under.
- Next Problem: Find a specific MT that maps a given generalized circle in a z-plane to a given generalized circle in a w-plane.
- Lines: Knowledge of two distinct points is enough to determine the equation of the line passing through those points.
- Circles: Three distinct points suffice.
- Generalized Circles: Knowledge of the MT of three points is enough to determine the formula of the transformation.

Example: mapping the generalized circles in z-plane onto the real axis in w-plane

Find MT which maps $z_1 \to w_1 = 0$, $z_2 \to w_2 = 1$, and $z_3 \to w_3 = \infty$.

Solution: Plugging the given mapping points into the MT, we get

$$\frac{az_1 + b}{cz_1 + d} = 0, \quad \frac{az_2 + b}{cz_2 + d} = 1, \quad \frac{az_3 + b}{cz_3 + d} = \infty.$$

Thus $b = -az_1$ and $d = -cz_3$, and the middle equation becomes

$$\frac{(z_2 - z_1)a}{(z_2 - z_3)c} = 1.$$

Choose $a = z_2 - z_3, c = z_2 - z_1$. Therefore

$$b = -az_1 = -z_1(z_2 - z_3), \quad d = -cz_3 = -z_3(z_2 - z_1).$$

Hence the required MT is

$$w = \frac{az + b}{cz + d} = \frac{(z_2 - z_3)z - z_1(z_2 - z_3)}{(z_2 - z_1)z - z_3(z_2 - z_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}.$$

Cross-Ratio Formula

Definition

The cross-ratio of the four points z, z_1, z_2, and z_3, is denoted by the ordered coordinates (z, z_1, z_2, z_3), that is,

$$(z, z_1, z_2, z_3) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}.$$

Theorem (Cross-Ratio Formula)

The MT which maps $z_1 \to w_1$, $z_2 \to w_2$, and $z_3 \to w_3$ is

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

which is the same as solving for w in terms of z from

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}.$$

Proof of Cross-Ratio Formula

Let $W = f(z) = (z, z_1, z_2, z_3)$ be the MT that maps the finite points z_1, z_2, and z_3 onto the points $W_1 = 0$, $W_2 = 1$, and $W_3 = \infty$, respectively. This mapping corresponds to the mapping of the generalzed circles in z-plane onto the real axis in W-plane. Also let $W = g(w) = (w, w_1, w_2, w_3)$ be the MT that maps the finite points w_1, w_2, and w_3 onto the points $W_1 = 0$, $W_2 = 1$, and $W_3 = \infty$, respectively. This mapping corresponds to the mapping the of generalzed circles in w-plane onto the real axis in W-plane. Hence

$$w = g^{-1}(W) = g^{-1}(f(z)),$$

which implies

$$g(w) = f(z).$$

This is equivalent to

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3).$$

Cross-Ratio Formula

- Solving for w in terms of z from

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

  gives the desired MT which maps $z_1 \longrightarrow w_1$, $z_2 \longrightarrow w_2$, and $z_3 \longrightarrow w_3$.
- If one of the z_i or w_i is ∞, the MT is obtained from the Cross-Ratio Formula by simply deleting the factors involving ∞.