UNIVERSITI TEKNOLOGI MALAYSIA

Chap 7: Residue Theory

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Outline:

- Residue Theorem
- Evaluation of Complex Integrals using Residue Theory
- Evaluation of Definite Trigonometric Integrals using Residue Theory
- Evaluation of Improper Real Integrals using Residue Theory

Introduction

- As noted in Chapter 5, there exist integrals which cannot be evaluated by means of Cauchy-Goursat theorem, Cauchy's integral fomula or the generalized Cauchy's integral formula.
- In this chapter, based on Laurent series, we shall develop a powerful technique for evaluating a larger class of complex integrals.
- We shall describe some applications of Cauchy Residue Theorem in the evaluations of definite real integrals of rational functions of sines and cosines.
- In scientific problems, one frequently encounters definite real integrals in which one (or both) of the limits is infinite. We shall next show how the theory of residues can be applied to evaluate the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ for certain functions f.

Residue Theorem

Definition

Suppose f has an isolated singularity at the point z_0. Then the coefficient a_{-1} of $(z - z_0)^{-1}$ in the Laurent series expansion of f around z_0 is called the **residue** of f at z_0 and is denoted by

$$a_{-1} = \mathrm{Res}(f; z_0).$$

Theorem (Residue Theorem)

Suppose $f(z)$ is analytic on and inside the positively oriented Jordan curve Γ except for a single isolated singularity, z_0, lying interior to Γ. Then

$$\int_\Gamma f(z)\, dz = 2\pi i\, \mathrm{Res}(f; z_0).$$

Calculation of Residues

Theorem (Residue Theorem)

Suppose $f(z)$ is analytic on and inside the positively oriented Jordan curve Γ except for a single isolated singularity, z_0, lying interior to Γ. Then

$$\int_{\Gamma} f(z)\,dz = 2\pi i\,\mathrm{Res}(f; z_0).$$

REMARK: One MUST obtain the Laurent series expansion first in order to determine the residue for the evaluation of the above integral. This requirement is true in general when dealing with essential singular points. For poles, the residues can be computed by special formulas, thus bypassing the need for Laurent expansion.

Calculation of Residues for Poles

Theorem

If f has a pole of order m at z_0, then

$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right].$$

In particular if z_0 is a simple pole $(m = 1)$ for f, then

$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Calculation of Residues for Simple Poles

Useful formula for computing residues at simple poles:

Theorem

Suppose $f(z) = P(z)/Q(z)$ *has a simple pole at* z_0 *such that*

(i) $P(z)$ *and* $Q(z)$ *are analytic at* $z = z_0$,

(ii) $Q(z)$ *has a simple zero at* z_0,

(iii) $P(z_0) \neq 0$.

Then

$$\operatorname{Res}(f; z_0) = \frac{P(z_0)}{Q'(z_0)}.$$

Generalized Residue Theorem

Like the Cauchy's Integral Formula, the Residue Theorem can also be generalized as follows.

Theorem (Cauchy's Residue Theorem)

Suppose $f(z)$ is analytic on and inside the positively oriented Jordan curve Γ except for a finite number of isolated singularities at the points $z_1, z_2, \ldots, z_n$, lying interior to Γ. Then

$$\int_{\Gamma} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f; z_k).$$

Remark: This theorem is also valid for multiply connected regions.

Evaluation of Definite Real Trigonometric Integrals

Theorem

Let $F(\cos\theta, \sin\theta)$ be a real function of $\cos\theta$ and $\sin\theta$ that is defined over $[0, 2\pi]$. If $z = e^{i\theta}$ and $C : |z| = 1$, then

$$\int_0^{2\pi} F(\cos\theta\ ,\ \sin\theta)\, d\theta = \int_C F\left[\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right] \cdot \frac{dz}{iz}.$$

If $F(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$, then the complex integrand in the above Theorem forms a rational function of z. Once its poles inside $|z| = 1$ have been determined, we can evaluate the complex integral by means of the Cauchy Residue Theorem.

Evaluation of Improper Real Integrals

Suppose $f(x)$ is continuous on $(-\infty, \infty)$. If the “symmetric limit"

$$\lim_{\rho \to \infty} \int_{-\rho}^{\rho} f(x)\, dx,$$

exists, it is called the **principal value integral** of f from $-\infty$ to ∞ and we write

$$\text{PV} \int_{-\infty}^{\infty} f(x)\, dx = \lim_{\rho \to \infty} \int_{-\rho}^{\rho} f(x)\, dx.$$

Evaluation of Principal Value Integrals

Theorem (A)

Suppose $f(x)$ is continuous on $(-\infty, \infty)$, such that

(i) *$f(z)$ is analytic on and above the real axis except for a finite number of isolated singularities $z_1, z_2, \ldots, z_n$ in the open upper half-plane $\operatorname{Im} z > 0$,*

(ii) $\displaystyle\lim_{\rho\to\infty} \int_{\Gamma_\rho} f(z)\, dz = 0$, *where* $\Gamma_\rho : z = \rho e^{it}, 0 \le t \le \pi$.

Then

$$\mathrm{PV} \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{k=1}^{n} \mathrm{Res}[f(z); z_k].$$

Fulfilment of Condition (ii)

The following theorem gives a class of functions for which the condition (ii) can be satisfied.

Theorem (B)

If $f(z) = P(z)/Q(z)$ *is a rational function such that*

$$\text{degree } Q - \text{degree } P \geq 2,$$

then

$$\lim_{\rho\to\infty} \int_{\Gamma_\rho} f(z)\,dz = 0,$$

where $\Gamma_\rho : z = \rho e^{it}, 0 \leq t \leq \pi$.

Evaluation of Principal Value Integrals

Theorem (A + B)

Suppose $f(z) = P(z)/Q(z)$ is a rational function with no poles on the real axis such that

$$\text{degree } Q - \text{degree } P \geq 2.$$

Then

$$\mathrm{PV} \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \mathrm{Res}[f(z); z_k].$$

where $z_1, z_2, \ldots, z_n$ are poles of $f(z) = P(z)/Q(z)$ that lie in the open upper half-plane $\mathrm{Im}\, z > 0$.

Another Fulfilment of Condition (ii)

The following theorem gives another class of functions for which the condition (ii) can be satisfied.

Theorem (Jordan's Lemma)

Suppose α is a non-zero real number and $P(z)/Q(z)$ is a rational function such that

$$\text{degree } Q - \text{degree } P \geq 1.$$

Then

$$\lim_{\rho\to\infty} \int_{\Gamma_\rho} e^{i\alpha z} \frac{P(z)}{Q(z)}\, dz = 0,$$

where $\Gamma_\rho : z = \rho e^{\operatorname{sgn}(\alpha) it}$, $0 \leq t \leq \pi$.

Evaluation of Fourier Integral Transform

The previous result enables the application of the residue theorem to improper real integrals of the form

$$\text{PV}\int_{-\infty}^{\infty} \cos(\alpha x)\frac{P(x)}{Q(x)}\,dx \quad \text{or} \quad \text{PV}\int_{-\infty}^{\infty} \sin(\alpha x)\frac{P(x)}{Q(x)}\,dx.$$

They must however be treated by writing them as the real and imaginary parts of

$$\text{PV}\int_{-\infty}^{\infty} e^{i\alpha x}\frac{P(x)}{Q(x)}\,dx.$$

Such integral is also known as the **Fourier integral transform** which forms an important tool for solving various partial differential equations of science and engineering.