Chap 6: Complex Series

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Chap 6: Complex Series

Outline:

- Convergence Tests
- Power Series
- Taylor Series
- Laurent Series
- Zeroes and Singularities of Analytic Functions

Overview of Chap 6

- In calculus, several convergence tests for real series have been studied: divergence test, integral test, comparison test, ratio test, root test.
- The notions of power series and Taylor series for representing real functions are also discussed in calculus.
- This chapter shall extend these ideas to series of complex numbers or functions of complex variables.
- The complex Taylor series for representing analytic functions has generalization to Laurent series which has no analogue in calculus.
- Based on Laurent series, several facts on the zeros and singularities of analytic functions can be established.
- Knowledge of Laurent series is necessary in developing a powerful technique for evaluating complex integrals (Chap 7).

Sequence of Complex Numbers

Definition

A sequence of complex numbers $\{z_n\}_{n=1}^{\infty} = z_1, z_2, z_3, \ldots$ is said to have a limit complex number A or **converge** to A, and written as

$$\lim_{n\to\infty} z_n = A \quad \text{or} \quad z_n \to A, \quad \text{when} \quad n \to \infty$$

if for any $\epsilon > 0$, there exists an integer N such that

$$|z_n - A| \leq \epsilon \quad \text{for all} \quad n \geq N.$$

A sequence that does not converge is said to be **divergent**.

Theorem on Limits of Sequences

Theorem

If $\lim z_n = A$ *and* $\lim w_n = B$*, then*

(a) $\lim(z_n \pm w_n) = \lim z_n \pm \lim w_n = A \pm B,$

(b) $\lim(z_n w_n) = (\lim z_n)(\lim w_n) = AB,$

(c) $\lim \dfrac{z_n}{w_n} = \dfrac{\lim z_n}{\lim w_n} = \dfrac{A}{B}$*, provided* $B \neq 0$.

The definition for sequence of complex numbers can be extended to sequence of functions of a complex variable as shown next.

Sequence of Complex Functions

Definition

Let $\{u_n(z)\}_{n=1}^{\infty} = u_1(z), u_2(z), u_3(z), \ldots$, denote a sequence of functions of z defined on a region Ω of the complex plane. The sequence $\{u_n(z)\}_{n=1}^{\infty}$ is said to have a limit complex function $U(z)$ or **converges** to U, and written as

$$\lim_{n\to\infty} u_n(z) = U(z) \quad \text{or} \quad u_n(z) \to U(z), \quad \text{when} \quad n \to \infty$$

if for any $\epsilon > 0$, there exists an integer $N(\epsilon, z)$ such that

$$|u_n(z) - U(z)| \leq \epsilon \quad \text{for all} \quad n \geq N.$$

A sequence that does not converge at some point $z \in \Omega$ is said to be **divergent** at z.

Uniform Convergence

- Recall the definition of convergence:

$$\lim_{n\to\infty} u_n(z) = U(z) \quad \text{or} \quad u_n(z) \to U(z), \quad \text{when} \quad n \to \infty$$

if for any $\epsilon > 0$, there exists an integer $N(\epsilon, z)$ such that

$$|u_n(z) - U(z)| \leq \epsilon \quad \text{for all} \quad n \geq N.$$

- If $N = N(\epsilon)$, i.e. N depends only on ϵ and not on $z \in \Omega$, we say that $\{u_n(z)\}_{n=1}^{\infty}$ **converges uniformly**, or is **uniformly convergent**, to $U(z)$ for all z in Ω.

Series of Complex Numbers

- The infinite series of complex numbers

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \cdots + z_n + \cdots$$

with $z_n = x_n + iy_n$, is said to be **convergent** if the sequence of partial sums

$$S_1 = z_1, \quad S_2 = z_1 + z_2, \quad S_3 = z_1 + z_2 + z_3, \quad \ldots$$

form a converging sequence.

- If $\lim_{n \to \infty} S_n = S$, then S is the called the **sum of the series** and we write

$$\sum_{n=1}^{\infty} z_n = S.$$

- If the sequence $\{S_n\}$ does not posses a limit, we say that the series $\sum_{n=1}^{\infty} z_n$ **diverges.**

Divergence Test & Geometric Series

Theorem (Divergence Test)

If $\lim_{n\to\infty} z_n$ *is different from zero (or does not exist), then* $\sum_{n=1}^{\infty} z_n$ *diverges.*

Theorem (Geometric Series Theorem)

Let r *be any complex number, and let* $c \neq 0$ *and* $m \geq 0$. *Then the geometric series* $\sum_{n=m}^{\infty} cr^n$ *converges if and only if* $|r| < 1$. *For* $|r| < 1$

$$\sum_{n=m}^{\infty} cr^n = \frac{cr^m}{1-r}.$$

Absolute Convergence Test

Theorem

If $\sum_{n=1}^{\infty} |z_n|$ converges, then $\sum_{n=1}^{\infty} z_n$ converges (absolutely).

Theorem

The infinite series $\sum_{n=1}^{\infty} z_n$ converges if and only if for any $\epsilon > 0$, there is an integer N such that

$$|z_{n+1} + z_{n+2} + \cdots + z_m| < \epsilon$$

for all $m \geq n \geq N$.

Ratio Test

Theorem (Ratio Test)

Let $\sum_{n=1}^{\infty} z_n$ be a series of complex numbers with $z_n \neq 0$ for $n \geq 1$ and that

$$\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = r \quad (\textit{possibly } \infty).$$

Then the series $\sum_{n=1}^{\infty} z_n$ converges if $r < 1$ and diverges if $r > 1$. If $r = 1$, then from this test alone we cannot draw any conclusion about the convergence of the series.

Root Test

Theorem (Root Test)

Let $\sum_{n=1}^{\infty} z_n$ be a series of complex numbers and that

$$\lim_{n\to\infty} \sqrt[n]{|z_n|} = r \quad (\textit{possibly } \infty).$$

Then the series $\sum_{n=1}^{\infty} z_n$ converges if $r < 1$ and diverges if $r > 1$. If $r = 1$, then from this test alone we cannot draw any conclusion about the convergence of the series.

Series of Complex Functions

The definition of series of complex numbers can be extended to series of functions of complex variable z. Let

$$\{u_n(z)\}_{n=1}^{\infty} = u_1(z), u_2(z), u_3(z), \dots,$$

denote a sequence of functions of z defined in a region Ω of the complex plane. The infinite series

$$\sum_{n=1}^{\infty} u_n(z) = u_1(z) + u_2(z) + u_3(z) + \cdots + u_n(z) + \cdots$$

is said to be **convergent** in Ω if the sequence of partial sums

$$S_1(z) = u_1(z), \quad S_2(z) = u_1(z)+u_2(z), \quad S_3(z) = u_1(z)+u_2(z)+u_3(z)$$

form a converging sequence in Ω. If $\lim_{n\to\infty} S_n(z) = f(z)$,then f is the called the **sum of the series** and we write

$$\sum_{n=1}^{\infty} u_n(z) = f(z).$$

Uniform Convergence and Weierstrass M-Test

- If the sequence of partial sums $\{S_n(z)\}$ converges uniformly in Ω, we say that the series $\sum_{n=1}^{\infty} u_n(z)$ **converges uniformly**, or is **uniformly convergent**, to $f(z)$ for all z in Ω.
- The following test is often adequate for proving the uniform convergence of some series.

Theorem (Weierstrass M-Test)

If $|u_n(z)| \leq M_n$ where M_n is a positive constant independent of z in a region Ω and $\sum_{n=0}^{\infty} M_n$ converges, then $\sum_{n=0}^{\infty} u_n(z)$ is uniformly convergent in Ω.

- By applying Weierstrass M-Test, it can be shown that the geometric series converges uniformly in every closed disk $|z| \leq \rho < 1$.

Benefits of Uniform Convergence

Theorem

Suppose $u_n(z)$ is analytic in a region Ω and $\sum_{n=1}^{\infty} u_n(z) = f(z)$ uniformly in Ω. Then $f(z)$ is analytic in Ω.

Theorem (Differentiation Term-by-Term)

Suppose $u_n(z)$ is continuously differentiable in a region Ω, $\sum_{n=1}^{\infty} u_n(z) = f(z)$ in Ω, and $\sum_{n=1}^{\infty} u_n'(z) = g(z)$ uniformly in Ω. Then

$$\frac{d}{dz}\sum_{n=1}^{\infty} u_n(z) = \sum_{n=1}^{\infty} u_n'(z) = f'(z) = g(z).$$

Another Benefit of Uniform Convergence

Theorem (Integration Term-by-Term)

Suppose $u_n(z)$ is continuous in a region Ω and $\sum_{n=1}^{\infty} u_n(z) = f(z)$ uniformly in Ω. Then $f(z)$ is continuous in Ω and

$$\int_\Gamma \left(\sum_{n=1}^{\infty} u_n(z) \right) dz = \sum_{n=1}^{\infty} \int_\Gamma u_n(z)\, dz = \int_\Gamma f(z)\, dz,$$

where Γ is a curve in Ω.

Power Series

The geometric series $\sum_{n=0}^{\infty} z^n$ is a special case of a general type of series known as the power series.

Definition

A series of the form $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, where z_0, a_n are complex constants, is called the power series. The constants a_n are called the coefficients of the power series.

Given a power series, several questions arise naturally:

- Does it converge?
- If so, where?
- Now for each fixed z, a power series is an infinite series of complex contants. So whatever we know about infinite series in the previous section applies.

Convergence & Divergence of Power Series

Theorem

For any power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, there exists a real number R ($0 \leq R \leq \infty$), which depends only on the coefficients a_n, such that one of the following cases holds:

(a) *The series converges for $|z-z_0| = R = 0$, i.e. only for $z = z_0$.*

(b) *The series converges for $|z-z_0| = R = \infty$, i.e. for all values of z.*

(c) *For $0 < R < \infty$, the series converges for $|z-z_0| < R$ and diverges for $|z-z_0| > R$, while for $|z-z_0| = R$ it may or may not converge.*

Geometrically the equation $|z-z_0| = R$, for $0 < R < \infty$, decribes a circle of radius R with center at $z = z_0$. For this reason, R is called the **radius of convergence** and the circle $|z-z_0| = R$ is called the **circle of convergence**.

Uniform Convergence of Power Series

By Weierstrass M-Test, one can established the uniform convergence of power series.

Theorem

Suppose the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ *has radius of convergence* $R > 0$. *If* $0 < \rho < R$*, then the power series converges uniformly and absolutely in* $|z - z_0| \leq \rho$.

- A power series converges uniformly and absolutely in any region entirely inside its circle of convergence.
- A power series can be differentiated and integrated term-by-term inside its circle of convergence, as long as the curve of integration lies inside the circle of convergence.

Power Series and Analytic Function

Theorem

The power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, *for* $|z - z_0| < R$, *is analytic in* $|z - z_0| < R$ *and its derivatives can be obtained by differentiating term-by-term, i.e.*

$$\sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1} = f'(z)$$

and this is valid in the same circle as the original series, i.e. $|z - z_0| < R$.

- A power series can be differentiated as many times as we like and all results in power series which converge in the same circle as the original series.
- Is there a formula between the coefficients a_n of the power series and its sum $f(z)$?

Taylor Series

Theorem (Taylor Series Theorem)

Suppose $f(z)$ is analytic in the region $|z - z_0| < R$. Then f has the Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \quad |z - z_0| < R.$$

The Taylor series converges to $f(z)$ for any z in the largest open disk centered at z_0 for which f is analytic.

- This formula clearly resembles the Taylor series studied in calculus.
- This shows that a converging power series is a Taylor series of its sum.

Taylor Series Expansions of Elementary Functions

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad |z| < \infty$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!}, \quad |z| < \infty$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots = \sum_{n=0}^{\infty} (-1)^{n} \frac{z^{2n}}{(2n)!}, \quad |z| < \infty$$

$$\operatorname{Log} z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(z-1)^n}{n}, \quad |z-1| < 1$$

Is Series Expansion About A Singular Point Possible?

- If f is analytic at z_0, then the series expansion of f analytic in the neighborhood of z_0 is the Taylor series.
- If we replaced the region of analyticity of f from the disk centered at z_0 to an annulus centered at z_0, then the analyticity of f at z_0 is immaterial. What then is the possible series expansion of f analytic in an annulus about z_0?
- The search of a series expansion of a function f about a point, which may or not be a singular point of f, leads to a generalization of the Taylor series known as the Laurent series due to Pierre Alphonse Laurent (1813-1854, French).

Laurent Series Expansion

Theorem (Laurent Series Theorem)

Let $0 \leq r < R \leq \infty$. If f is analytic in the annulus $r < |z - z_0| < R$, then f has the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad r < |z - z_0| < R,$$

where

$$a_n = \frac{1}{2\pi i} \int_{|w - z_0| = \rho} \frac{f(w)}{(w - z_0)^{n+1}}\, dw, \quad n = 0 \pm 1 \pm 2, \ldots$$

for $r < \rho < R$.

Remarks on Laurent Series

- The Laurent series includes the Taylor series as a special case.
- Knowledge of Laurent series is necessary for our discussion on residue theory for evaluating integrals in Chapter 7.
- Sometimes the coefficients a_n in the Laurent series can be calculated by manipulation of known series and partial fractions.
- The Laurent series may contain an infinite number of both negative and positive powers.
- In some cases, the Laurent series may contain a finite number of negative powers but infinite number of positive powers.
- Yet in another instance, a Laurent series may contain an infinite number of negative powers but finite number of positive powers.

Zeros and Singularities of Analytic Functions

Based on Taylor series and Laurent series, several facts on the zeros and singularities of analytic functions can be established.

Definition

A point z_0 is called a **zero of order** m for f if f is analytic at z_0 and

$$f(z_0) = \cdots = f^{(m-1)}(z_0) = 0, f^{(m)}(z_0) \neq 0.$$

If $m = 1$, then z_0 is called a **simple zero** for f.

Theorem

A function f has a zero of order m if and only if

$$f(z) = (z - z_0)^m g(z),$$

where g is analytic at z_0 and $g(z_0) \neq 0$.

Singularities of Analytic Functions

Suppose $f(z)$ is not analytic at z_0 but analytic in the puntured disk $0 < |z - z_0| < R$. Thus f has an isolated singularity at z_0. Recall that the Laurent series expansion of a function f around z_0 may contains both negative and positive powers of $(z - z_0)$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n = \sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n} + \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

Hence the Laurent series is decomposed into two parts:

(a) The series $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ which is analytic in $|z - z_0| > 0$. This series which contains only the negative powers of $z - z_0$ is called the **principal part** of f at $z = z_0$.

(b) The series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ which is analytic in $|z - z_0| < R$. This series which contains only the nonnegative powers of $z - z_0$ is called the **regular part** of f at $z = z_0$.

Three Types of Isolated Singularity

The singularity of f at z_0 is clearly reflected in the principal part. The principal part has three possibilities which gives rise to three types of isolated singularity of f at z_0:

(a) The principal part did not appear, i.e. $a_{-n} = 0$ for $n = 1, 2, 3, \ldots$. In this case, we say z_0 is a **removable singularity** for f.

(b) The principal part has a finite number of negative powers of $z - z_0$, i.e. $a_{-m} \neq 0$ for some positive integer m but $a_{-n} = 0$ for all $n > m$. In this case, we say z_0 is a **pole of order** m for f. If $m = 1$, we called z_0 as a **simple pole** for f.

(c) The principal part has an infinite number of negative powers of $z - z_0$, i.e. $a_{-n} \neq 0$ for an infinite number positive integer values of n. In this case, we say z_0 is an **essential singularity** for f.

Theorem

Suppose f has an isolated singularity at z_0. Then

(a) z_0 *is a removable singularity* $\Leftrightarrow$ $|f|$ *is bounder near* z_0 $\Leftrightarrow$ $\lim_{z\to z_0} f(z)$ *exists* $\Leftrightarrow$ f *can be redefined at* z_0 *so that* f *is analytic at* z_0.

(b) z_0 *is a pole of order* m $\Leftrightarrow$ $|f|$ *is unbounded near* z_0 $\Leftrightarrow$ $\lim_{z\to z_0} f(z) = \infty$ $\Leftrightarrow$ f *can be rewritten as* $f(z) = g(z)/(z - z_0)^m$ *where* g *is analytic at* z_0 *with* $g(z_0) \neq 0$.

(c) z_0 *is an essential singularity* $\Leftrightarrow$ $|f|$ *neither is bounded near* z_0 *nor goes to infinity as* $z \to z_0$.

The following theorem relates the concepts of zeros and poles of a function with its reciprocal.

Theorem

If f has a zero of order m at z_0, then $1/f$ has a pole of order m at z_0. If f has a pole of order m at z_0, then $1/f$ has a zero of order m at z_0. If f has a removable singularity at z_0, then $1/f$ also has a removable singularity at z_0.