Chap 5: Complex Integration

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Chap 5: Complex Integration

Outline:

- Curves on the Complex Plane
- Integration on the Complex Plane
- Fundamental Theorem
- Cauchy-Goursat Theorem
- Cauchy's Integral Formula

Complex Integration

A complex integral is a generalization of a real integral studied in calculus in two ways:

- the function being integrated is a complex function,
- and the integration acts on arcs or closed curves.

Many problems in mathematics, physics, and statistics involve real integrals in the form of

$$\int_0^{2\pi} R(\sin\theta, \cos\theta)\, d\theta, \quad \int_k^{\infty} f(x)\, dx, \quad \int_{-\infty}^{k} f(x)\, dx, \quad \int_{-\infty}^{\infty} f(x)\, dx.$$

These integrals can be solved efficiently using the notion of complex integrals.

Complex Integral and Dirichlet Problem

Let Ω be a simply connected region in the complex plane with boundary Γ, and let ϕ be a continuous real-valued function on Γ. The Dirichlet problem consists in finding a function u satisfying the conditions:

1. u is continuous in $\Omega \cup \Gamma$.
2. u is harmonic in Ω.
3. $u = \phi$ on Γ.

It can be shown that the function u has the form

$$u(z) = \frac{1}{2\pi i} \int_{\Gamma} \phi(w) \frac{1 - |f(z)|^2}{|f(w) - f(z)|^2} \frac{f'(w)}{f(w)}\, dw, \quad z \in \Omega,$$

where f is a one-to-one analytic function that maps Ω onto a unit disk. The integral in the formula above is a complex integral.

Curves on the Complex Plane

A complex integral is an integral of a complex function over a certain curve on a complex plane.

Definition

A **curve** Γ on the complex plane is defined by a complex-valued function $z(t)$ on the real interval $a \leq t \leq b$ as

$$\Gamma : z = z(t) = x(t) + iy(t), \quad a \leq t \leq b, \quad b > a.$$

The real variable t is called the **parameter for curve**, and the representation above is called the **parametric representation** of Γ. The **initial point** of Γ is $z(a) = x(a) + iy(a)$ and the **terminal point** of Γ is $z(b) = x(b) + iy(b)$.

Thus a given curve on the complex plane has a specific orientation, i.e. a directed curve.

Curves on the Complex Plane

Definition

A curve Γ is **closed** if $z(a) = z(b)$.

A curve that is not closed is also known as an **arc**.

Definition

A curve Γ is **simple** if $z(t_1) \neq z(t_2)$ for $a \leq t_1 < t_2 < b$.

Geometrically speaking, a simple curve does not cross.

Definition

A curve Γ is **continuous** if the complex valued function $z(t)$ is continuous with respect to t.

Curves on the Complex Plane

Definition

A curve Γ is **differentiable** if $z'(t)$ exists for $a \leq t \leq b$.

Definition

A curve Γ is said to be **smooth** if $z'(t)$ is continuous and $z'(t) \neq 0$ for $a \leq t \leq b$.

Geometrically, a tangent on a smooth curve is well defined and it changes continuously. Thus a smooth curve does not contain any corners or cusps. The term **regular** is sometimes used in place of smooth.

Definition

A curve Γ said to be **piecewise smooth**, if it is a joint of several smooth curves.

Curves on the Complex Plane

Definition

If $z = z(t), a \leq t \leq b$ parametrizes Γ, then the curve $-\Gamma$ is defined as

$$-\Gamma : z = z(-t), \quad -b \leq t \leq -a$$

or

$$-\Gamma : z = z(a + b - t), \quad a \leq t \leq b.$$

The curve $-\Gamma$ is similar to the curve Γ but has the opposite direction.

Definition

A closed simple curve is also known as a **Jordan curve** or a **loop**.

Curves on the Complex Plane

Definition

A Jordan curve Γ is said to be in a **positive orientation** if the domain bounded by Γ is on the left when the curve is traversed in its direction.

From calculus, a curve on the Cartesian plane may be defined by the parametric representation $x = x(t), y = y(t), t_0 \leq t \leq t_1$. If the graph of this equation sits on the complex plane, then the parametrization is given by

$$z(t) = x(t) + i\,y(t), \quad t_0 \leq t \leq t_1. \tag{1}$$

Since the parametric representation of a curve on the Cartesian plane is not unique, the curve on the complex plane also has many possible parametric representations.

Curves on the Complex Plane

In general a circle with the Cartesian equation of the form

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

has the parametric representation

$$z(t) = z_0 + re^{it}, \quad 0 \le t \le 2\pi, \tag{2}$$

where $z_0 = x_0 + iy_0$. Note that this equation is equivalent to

$$|z - z_0| = r$$

which is a complex equation for the circle with centre z_0 and radius r. The parametrization above describes a circle in a counterclockwise direction. For a clockwise direction, the parametrization is

$$z(t) = z_0 + re^{-it}, \quad -2\pi \le t \le 0. \tag{3}$$

Curves on the Complex Plane

In vector calculus, we learnt that a parametric representation for the line through two distinct points $\mathbf{u_0}$ and $\mathbf{u_1}$ is given by

$$\mathbf{u} = (\mathbf{1} - \mathbf{t})\mathbf{u_0} + \mathbf{t}\mathbf{u_1} = \mathbf{u_0} + \mathbf{t}(\mathbf{u_1} - \mathbf{u_0}), \quad \mathbf{0} \leq \mathbf{t} \leq \mathbf{1}.$$

Since complex numbers can be regarded as vectors, a parametric representation for the line through two complex distinct points z_0 and z_1 is given by

$$z(t) = (1 - t)z_0 + tz_1 = z_0 + t(z_1 - z_0), \quad 0 \leq t \leq 1. \qquad (4)$$

with the initial point z_0 and the end point z_1.

Integration on the Complex Plane

In calculus, the concept of integration is usually introduced through the problem of finding area under a curve.
Consider a function f defined and continuous on the interval $a \leq x \leq b$. Divide this interval into n subintervals by introducing the points x_k, $k = 1, 2, \dots, n-1$, such that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Let $\triangle x_k$ be defined as

$$\triangle x_k = x_k - x_{k-1}, \quad k = 1, 2, \dots, n-1,$$

and let x_k^* denote any representative points satisfying

$$x_{k-1} \leq x_k^* \leq x_k, \quad k = 1, 2, \dots, n-1.$$

The definite integral of f from $x = a$ to $x = b$ is defined by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\triangle x_k$$

Integration on the Complex Plane

The approach just described for defining the definite integral $\int_a^b f(x)\,dx$ is also used in the same manner in defining a complex integral.

Definition

Let $f(z)$ be a complex function defined and continuous on the directed curve Γ on the complex plane. Let $z_k \in \Gamma$, $k = 0, 1, \ldots, n$, such that the point z_{k-1} precedes z_k for $k = 1, 2, \ldots, n$. Let z_k^* be a point on the subcurve from z_{k-1} to z_k for $k = 1, 2, \ldots, n$. If $\Delta z_k = z_k - z_{k-1}$, define

$$\int_\Gamma f(z)\,dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(z_k^*)\Delta z_k,$$

provided the limit exists. In this case, we say f is **integrable on the curve** Γ.

Properties of Complex Integration

Several properties of definite integrals in calculus also remain valid for complex integrals.

Theorem

Suppose f and g are integrable on the curve Γ and c be any complex constant. Then

(a) $\displaystyle\int_\Gamma [f(z) \pm g(z)]\,dz = \int_\Gamma f(z)\,dz \pm \int_\Gamma g(z)\,dz$

(b) $\displaystyle\int_\Gamma c\,f(z)\,dz = c\int_\Gamma f(z)\,dz$

(c) $\displaystyle\int_{-\Gamma} f(z)\,dz = -\int_\Gamma f(z)\,dz$

Parametrization Method for Complex Integration

Computing a complex integral by its definition is no easy task since it involves computing limit of a sum. But if the curve Γ admits a suitable parametrization, the problem of evaluating complex integrals reduces to evaluating real integrals.

Theorem

Suppose $\Gamma : z = z(t) = x(t) + iy(t)$, $\alpha \leq t \leq \beta$, *is a smooth curve and* $f(z) = u(z) + iv(z) = u(x, y) + iv(x, y)$ *is continuous on* Γ. *Then*

$$\int_{\Gamma} f(z)\,dz = \int_{\alpha}^{\beta} f(z(t))z'(t)\,dt.$$

Parametrization Independence

A complex integral is independent of any parametrization.

Theorem

Suppose f is continuous on the smooth curve Γ. Let $z = z_1(t)$, $a \leq t \leq b$, and $z = z_2(t)$, $c \leq t \leq d$, be two different parametrizations for Γ. Then

$$\int_\Gamma f(z)\,dz = \int_a^b f(z_1(t))z_1'(t)\,dt = \int_c^d f(z_2(t))z_2'(t)\,dt.$$

Integration on a Piecewise Smooth Curve

Integrals along a piecewise smooth curve are evaluated as in the following definition.

Definition

Suppose Γ is a piecewise smooth curve consisting of directed smooth curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$. We write $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$. Assume f is continuous on Γ. Then the complex integral of f over Γ is defined by

$$\int_\Gamma f(z)\,dz = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz + \cdots + \int_{\Gamma_n} f(z)\,dz.$$

ML Inequality

A good upper bound estimate of the absolute value of the complex integral is sometimes useful. For real integrals, if $f(x)$ is continuous on $[a, b]$ and $|f(x)| \leq M$ for all $x \in [a, b]$,we have

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx \leq M(b-a).$$

The analogous result for complex integral is as follows.

Theorem (ML Inequality)

Suppose f is continuous on the curve Γ and assume $|f(z)| \leq M$ for all $z \in \Gamma$. Then

$$\left| \int_\Gamma f(z)\, dz \right| \leq M\, L(\Gamma),$$

where $L(\Gamma)$ is the arc length of Γ.

Fundamental Theorem of Calculus

- In calculus, the concepts of derivative and integral is beautifully tied up in the Fundamental Theorem of Calculus: Let f be continuous on $[a, b]$ and suppose F is any antiderivative of f on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

- This theorem is perhaps the most important one in the study of calculus because it bridged together two dissimilar notions of derivative and integral, which arose from apparently unrelated problems (e.g. tangents and areas).
- This theorem has found a generalization in the theory of complex integral given next.

Fundamental Theorem of Complex Integration

Theorem (Fundamental Theorem)

If $F(z)$ is analytic with continuous derivative $F'(z) = f(z)$ on a region Ω that contains the piecewise smooth curve $\Gamma : z = z(t)$, $\alpha \leq t \leq \beta$, then

$$\int_\Gamma f(z)\,dz = F(z(\beta)) - F(z(\alpha)).$$

Observe that the evaluation of the integral above, subject to the condition stated, only depends on the antiderivative F of f evaluated at the end points of Γ. Hence the sometimes cumbersome parametrization process of Γ can be totally avoided, provided that an antiderivative of the integrand is known.

Cauchy-Goursat Theorem and Path Independence

In this section we shall discuss some results concerning complex integral which have no analogues with the real integral in calculus. First recall the following theorem studied in calculus of several variables.

Theorem (Green's Theorem)

Suppose Ω is a simply connected region bounded by the curve Γ. Assume $M(x, y)$, $N(x, y)$, $M_x(x, y)$, $M_y(x, y)$, $N_x(x, y)$, $N_y(x, y)$, are continuous over $\Omega \cup \Gamma$. Then

$$\int_\Gamma M(x, y)\, dx + N(x, y)\, dy = \iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy.$$

Cauchy's Theorem

Augustin Louis Cauchy (1789-1857, French) in the year 1814 made used of Green's theorem to derive the following result.

Theorem (Cauchy's Theorem)

Suppose $f(z)$ is analytic inside and on the piecewise Jordan curve Γ and $f'(z)$ is continuous on the same region. Then

$$\int_\Gamma f(z)\,dz = 0.$$

Later in 1900 Edward Goursat (1858-1936, French) proved the same result above but without any need for continuity condition on $f'(z)$. It will be established that if $f(z)$ is analytic over the region G, then so is $f'(z)$.

Cauchy-Goursat Theorem

Theorem (Cauchy's Theorem)

Suppose $f(z)$ is analytic inside and on the piecewise Jordan curve Γ and $f'(z)$ is continuous on the same region. Then

$$\int_{\Gamma} f(z)\,dz = 0.$$

Theorem (Cauchy-Goursat Theorem)

Let $f(z)$ be analytic inside and on the piecewise Jordan curve Γ. Then

$$\int_{\Gamma} f(z)\,dz = 0.$$

The Cauchy-Goursat theorem is often called Cauchy's integral theorem or briefly Cauchy's theorem.

Path Independence

Cauchy-Goursat theorem leads to various remarkable results related to complex integration that have many useful applications. One such result is path independece of integral with analytic function as its integrand.

Theorem (Path Independence)

Suppose $f(z)$ is analytic in a simply connected region Ω. If Γ_1 and Γ_2 are two different curves entirely in Ω sharing the same initial and terminal points , then

$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz.$$

CGT for Multiply Connected Region

- Cauchy-Goursat theorem is also valid for multiply-connected regions, i.e. regions which geometrically contain holes.
- A simply-connected region has no holes in it and has only one boundary.
- A doubly-connected region has one hole and has two boundaries.
- A doubly-connected region can be converted into a simply-connected region by introducing a crosscut, i.e. a simple arc connecting the two boundaries.
- In general, an n-connected region has $n - 1$ holes in it and has n boundaries.
- By introducing suitable $n - 1$ crosscuts, any multiply-connected region can be transformed into simply-connected region.

Loop Deformation Theorem

Another consequence of Cauchy-Goursat theorem is the possibilty of replacing complicated closed curves with more familiar ones for the purpose of integrating analytic functions over specified regions.

Theorem (Loop Deformation Theorem)

Suppose $f(z)$ is analytic in a region Ω. If Γ_1 and Γ_2 are two loops contained in Ω such that one can be continuously deformed into another without crossing a singularity of f, then

$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz.$$

Cauchy's Integral Formula

Theorem (Cauchy's Integral Formula)

Suppose Γ is a loop in a counterclockwise direction with a any point inside Γ. If f is analytic inside and on the loop Γ, then

$$\int_\Gamma \frac{f(z)}{z-a}\,dz = 2\pi i f(z)\Big|_{z=a} = 2\pi i f(a). \tag{5}$$

If we replaced a and z by z and w respectively in equation (5), we get the alternative form of Cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}\,dw. \tag{6}$$

The above result is truly remarkable. Under a simple hypothesis of f, the value of the function f at z, given by $f(z)$, is completely determined by the values of f on the loop Γ, $f(w)$.

Generalized Cauchy's Integral Formula

The Cauchy's integral formula can further be generalized to give the Generalized Cauchy's Integral Formula (GCIF) as follows.

Theorem (GCIF)

Suppose Γ is a loop in a counterclockwise direction with a any point inside Γ. If f is analytic inside and on the loop Γ, then

$$\int_\Gamma \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z)\Big|_{z=a} = \frac{2\pi i}{n!} f^{(n)}(a), \tag{7}$$

for $n = 1, 2, 3, \ldots$.

Interesting Interpretation of GCIF

Theorem (GCIF)

Suppose Γ is a loop in a counterclockwise direction with a any point inside Γ. If f is analytic inside and on the loop Γ, then

$$\int_{\Gamma} \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(z)\Big|_{z=a} = \frac{2\pi i}{n!} f^{(n)}(a), \qquad (8)$$

for $n = 1, 2, 3, \ldots$.

GCIF can also be alternatively written as

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w-z)^{n+1}}\, dw, \quad n = 1, 2, 3, \ldots \qquad (9)$$

This result is also truly remarkable. The theorem means that if f is analytic in a simply connected region Ω, then its derivatives $f', f'', \ldots, f^{(n)}$ are guaranteed to exist and analytic in Ω.

Limitations of CGT, CIF, GCIF

- Are there examples of complex integrals that could not be evaluated by means of Cauchy-Goursat theorem, Cauchy's integral fomula or the generalized Cauchy's integral fomula?
- Plenty! Some examples are

$$\int_{|z|=2} z\sin\left(\frac{1}{z}\right)\,dz, \quad \int_{|z|=1} e^{\frac{1}{z}}\,dz, \quad \int_{|z|=3} \cot z\,dz.$$

- Chapter 7 develops a powerful technique known as the **Residue Theorem** for the evaluation of these integrals.
- This technique requires knowledge of Laurent series, which is a series of complex functions (Chapter 6).