Chapter 3: Functions of Complex Variables

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Chap 3: Functions of Complex Variables

Outline:

- Definition of function of a complex variables
- Graphing complex functions
- Limits of complex functions
- Continuity of complex functions
- Derivative of complex functions
- L'Hopital's Rule
- Cauchy-Riemann equations
- Harmonic functions
- Harmonic conjugate

Definition of a Function of a Complex Variable

- A *function of a complex variable* f is a rule that associates each complex number z in a domain with one and only one complex number w.
- Usually the rule is written as

$$w = f(z).$$

- The collection of all possible values w for f is called the *range* of f.
- If $w = u + iv$, $z = x + iy$, then the above equation may be written in the form

$$u + iv = f(x + iy).$$

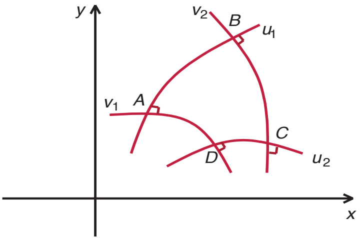
  This shows that u and v are real-valued functions of two variables x and y.

Graphing Complex Functions

- In general, graphing a complex function is impossible since it requires four dimensions; two dimensions for its domain, and two dimensions for its range.
- However a commonly used method to represent a complex function $w = f(z)$ geometrically, is by considering two complex planes; one for the domain of definition of f, i.e., the z-plane, and another for the range of f, called the *w-plane*
- Thus a complex function can be viewed geometrically as a mapping or transformation of one planar region onto another planar region.

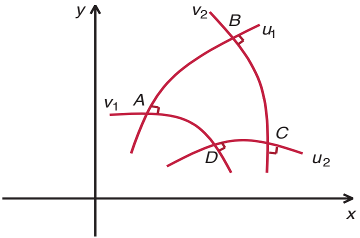
Mapping of Complex Function

The original region in z−plane

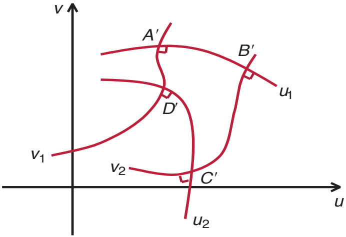


Mapping of Complex Function

The original region in $z-$plane

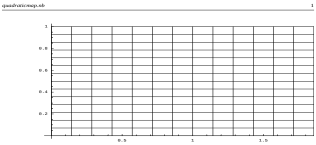


The transformed region in $w-$plane



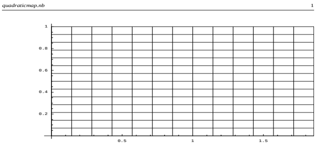
Mapping of $f(z) = z^2$

The original region in $z-$plane

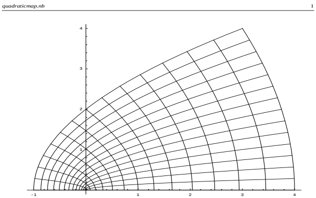


Mapping of $f(z) = z^2$

The original region in $z-$plane

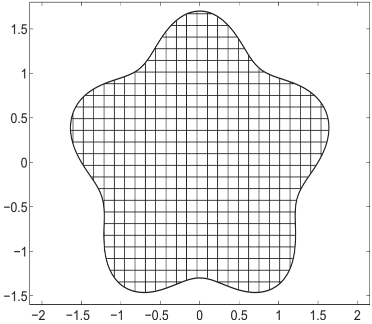


The transformed region in $w-$plane



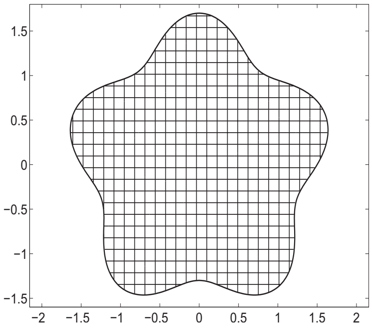
Mapping of Complex Function

The original region in $z-$plane

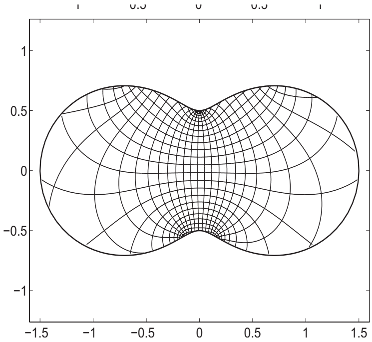


Mapping of Complex Function

The original region in $z-$plane



The transformed region in $w-$plane



Applications to Image Processing

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Applications to Image Processing

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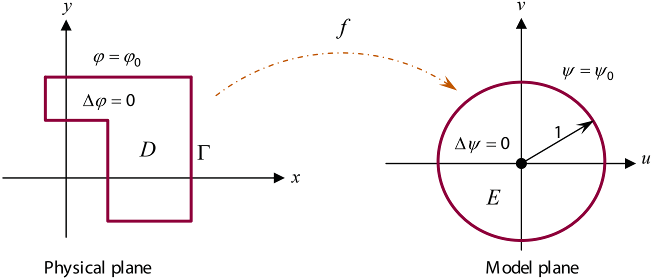
Application of Complex Mapping

Solving Dirichlet Problem



Limits of Complex Functions

As in calculus, the concept of complex limit plays a vital role in order to develop concepts of complex differentiation and complex integration. The definition of a limit of a complex function is similar to the one used in calculus.

Definition

Let f be a function defined in some neighborhood of z_0 with the possible exception of the point z_0 itself. The expression $\lim_{z \to z_0} f(z) = L$ or $f(z) \to L$ whenever $z \to z_0$ means that the funtion f has the limit L at the point z_0, if given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\textit{if } |z - z_0| < \delta, \textit{then } |f(z) - L| < \epsilon.$$

Limits of Complex Functions

$$\lim_{z \to z_0} f(z) = L$$

- The symbol $z \to z_0$ means that z may approach z_0 in ANY direction.
- If $f(z)$ has two different limits when $z \to z_0$ along two different paths, then we say that the limit $f(z)$ does not exist as $z \to z_0$.

Theorem on Limits

The following theorem on limits of complex functions is derived in exactly the same way as it is done in calculus.

Theorem

Assume

$$\lim_{z \to z_0} f(z) = L, \qquad \lim_{z \to z_0} g(z) = M.$$

Then

1. $\lim_{z \to z_0} [f(z) \pm g(z)] = L \pm M.$
2. $\lim_{z \to z_0} f(z)\, g(z) = L\, M.$
3. $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{L}{M}, \qquad (M \neq 0).$

Continuity of Complex Functions

Definition

A complex function f is said to be **continuous** at the point z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

If f is continuous at every point in the set S, then f is called **continuous on** S.

Theorem

Suppose that f and g are continuous at z_0. Then $f \pm g$ and fg are also continuous at z_0. If $g(z_0) \neq 0$, then f/g is also continuous at z_0. If h is continuous at $f(z_0)$, then $(h \circ f)(z) = h(f(z))$ is also continuous at z_0.

This theorem is proven analagously as in calculus.

Derivative of Complex Functions

Recall from calculus, that the derivative of $f(x)$ at x_0 is defined by

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

provided the limit exists. The derivative of a complex function is just a simple extension of the one above.

Definition

A complex function f defined for all z in a domain D is said to be **differentiable** at the point $z = z_0 \in D$ if the following limit exists:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

If the limit exists, it is called the **derivative** of f at z_0 and is denoted by $f'(z_0)$ or $\left.\dfrac{d}{dz} f(z)\right|_{z=z_0}$.

- Other forms are

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

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- The limit must tend to a unique complex $f'(z_0)$ regardless as to how z approaches z_0.

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- If f is differentiable in a δ-neighborhood of the point z_0, i.e., $|z-z_0|<\delta$, then f is said to be **analytic at the point** z_0.

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- A complex function that is analytic on the entire (or whole) complex plane is called an **entire function.**

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- If f is differentiable at all points of D, then f is said to be **analytic on** D.
- A complex function that is analytic on the entire (or whole) complex plane is called an **entire function.**
- It can be shown that for any constant c and any integer n,

$$\frac{d}{dz}(c) = 0, \quad \frac{d}{dz}(cz^n) = nz^{n-1}.$$

Theorem on Derivatives

Theorem

Suppose f and g are differentiable at z, and h is differentiable at $f(z)$. Then

1. $[f(z) \pm g(z)]' = f'(z) \pm g'(z)$
2. $[f(z)g(z)]' = f'(z)g(z) + f(z)g'(z)$
3. $\left[\dfrac{f(z)}{g(z)}\right]' = \dfrac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}, \quad g(z) \neq 0$
4. $[h(f(z))]' = h'(f(z)) \cdot f'(z)$ *(The chain rule)*

- The polynomial function

$$p(z) = a_n z^n + \cdots + a_2 z^2 + a_1 z + a_0$$

is therefore an entire function and

$$p'(z) = n a_n z^{n-1} + \cdots + 2 a_2 z + a_1 .$$

- If $w = f(z)$ and $W = g(w)$, then $W = g(f(z))$ and the chain rule implies

$$\frac{dW}{dz} = \frac{dW}{dw} \cdot \frac{dw}{dz} .$$

Higher Derivatives

The derivative $f'(z)$, also called the **first derivative** of $f(z)$, is itself a function. As in calculus, second, third, and higher derivatives are defined similarly by repeated differentiation:

$$\frac{d}{dz}f'(z) = f''(z) = \frac{d^2}{dz^2}f(z),$$
$$\frac{d}{dz}f''(z) = f'''(z) = \frac{d^3}{dz^3}f(z),$$
$$f^{(n)}(z) = \frac{d^n}{dz^n}f(z).$$

L'Hôpital's Rule

In calculus, one applies derivatives through L'Hôpital's rule to evaluate limits of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where $f(a) = g(a) = 0$. The complex version of L'Hôpital's rule is as follows.

Theorem (L'Hôpital's rule)

If $g(z_0) = 0$, $h(z_0) = 0$, and if $g(z)$ and $f(z)$ are differentiables at z_0 with $h'(z_0) \neq 0$, then

$$\lim_{z \to z_0} \frac{g(z)}{h(z)} = \frac{g'(z_0)}{h'(z_0)}.$$

Differentiability implies Continuity

The following result, familiar in calculus, is also valid in the complex case.

Theorem

If f is differentiable at z_0, then f is continuous at z_0.

The Cauchy-Riemann Equations

The following theorem shows an important consequence of differentiable functions.

Theorem

Suppose $z = x + iy$ *and* $f(z) = u(x, y) + iv(x, y)$. *If* $f'(z)$ *exists, then* u *and* v *satisfy the* **Cauchy-Riemann equations**

$$u_x(x, y) = v_y(x, y), \quad u_y(x, y) = -v_x(x, y).$$

- The Cauchy-Riemann equations will be briefly refered as the C-R equations.
- If C-R equations are not simultaneously satisfied at (x_0, y_0), then $f'(x_0 + iy_0)$ does not exist.
- Are the C-R equations sufficient to guarantee differentiability?

Necessary and Sufficient Conditions for Existence of $f'(z)$

The next theorem shows that if u, v, u_x, u_y, v_x, v_y satisfy certain continuity condition, then the C-R equations are not only necessary but also sufficient for the derivative of $f(z)$ to exist.

Theorem

If u, v, together with the first partial derivatives u_x, u_y, v_x, v_y, are all continuous in a domain D, then the C-R equations

$$u_x(x, y) = v_y(x, y), \quad u_y(x, y) = -v_x(x, y).$$

are necessary and sufficient conditions for $f(z) = u(x, y) + iv(x, y)$ to be analytic in D. The derivative of $f(z)$ is given either by

$$f'(z) = u_x + iv_x \quad \text{or} \quad f'(z) = v_y - iu_y.$$

Analyticity of Combinations of Analytic Functions

Results on the continuity and differentiability of combinations of functions have been given. Results on the analyticity of combinations of analytic functions are given next.

Theorem

If two functions are analytic on the same domain D, then their addition, difference, product, and composition are also analytic on D. The division of the two functions is also analytic on D except at points where the denominator becomes zero.

Harmonic Functions

There exists a classical relationship between analytic functions and harmonic functions, which gives rise to several applications of complex variables as a tool for modelling phenomenon of the physical world. We first review the notion of harmonic functions, familiar to many scientists and engineers.

Definition (Harmonic Function)

A real-valued function $\phi(x, y)$ satisfying the Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

is called a **harmonic function.**

There exists various problems in physics and mechanics involving Laplace's equation, particularly in fluid dynamics, hydrodynamics, jet flow, heat flow, and electrostatics.

Relationship Between Analytic Functions and Harmonic Functions

Theorem

If $f = u + iv$ is analytic in a domain D, with u and v having continuous first and second partial derivatives, then u and v are both harmonic in D.

The functions u and v from the analytic function $f = u + iv$ have an interesting geometrical interpretation. If we graph the level curves $u(x, y) = c$ and $v(x, y) = k$, where c and k are contants, then the level curves intersect at right angle.

Level Curves $u(x, y) = c$

example1.nb

```
f[z_] := z^2

g1 = ContourPlot[Re[f[x + I y]], {x, -1, 1}, {y, -1, 1}, ContourShading → Fal
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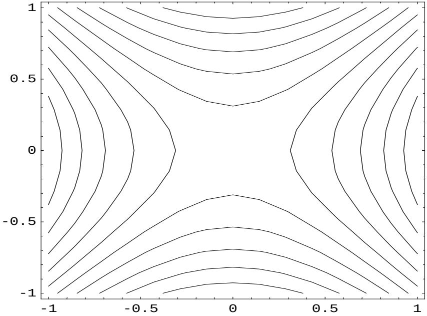

Level Curves $v(x, y) = k$

example1.nb

```
g2 = ContourPlot[Im[f[x + I y]], {x, -1, 1}, {y, -1, 1}, ContourShading → Fal
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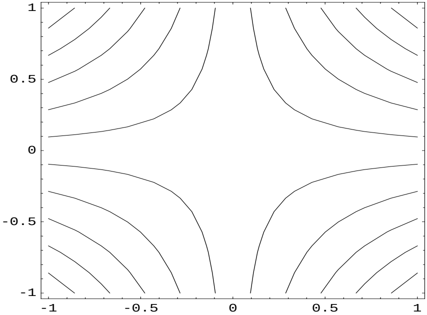
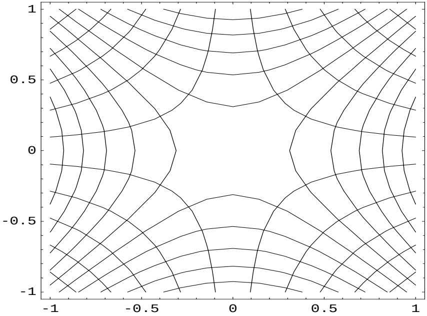


Intersection of Level Curves $u(x, y) = c$ and $v(x, y) = k$

example1.nb

Show[g1, g2]



- Graphics -

Harmonic Conjugate

- Given an analytic function $f = u + iv$, then each of u and v is necessarily harmonic.
- Suppose now that we are given a harmonic function u. Is it possible to determine a harmonic function $v(x, y)$ such that $f(z) = u + iv$ forms an analytic function?
- Such a function v is called a **harmonic conjugate** of u and can be constructed by means of the C-R equations.