Chapter 4: Elementary Functions of Complex Variables

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Outline:

- Complex Exponential Function
- Complex Trigonometric Functions
- Complex Hyperbolic Functions
- Complex Logarithmic Functions

Elementary Complex Functions

In calculus, several derivative formulas have been established for the elementary functions of real variables, such as the exponential, trigonometric, logarithm, hyperbolic, and the inverse functions. In this chapter we shall extend the definitions of the elementary functions from real variables to complex variables and obtain derivative formulas for them. We begin with the construction of a suitable definition for the complex exponential function, which forms a basis for defining other elementary functions of complex variables.

Complex exponential function

We wish to construct the complex exponential function e^z that would retain as many properties as in the real case. Thus we certainly want e^z to satisfy the following three notable properties:

(a) e^z reduces to e^x when $\operatorname{Im} z = 0$.

(b) $e^z\, e^w = e^{z+w}$

(c) e^z is analytic.

(d) $\dfrac{d}{dz} e^z = e^z$.

Definition of Complex Exponential Function

Definition

If $z = x + iy$, then the complex exponential function e^z is defined by

$$e^z = e^x(\cos y + i \sin y).$$

- e^z has the polar form $r(\cos\theta + i\sin\theta)$ representation. Hence

$$|e^z| = e^x, \qquad \arg e^z = y + 2k\pi,$$

  with k integers.
- Since $e^x > 0$, the function $e^z \neq 0$ for all z.
- If in the definition of e^z, we set $x = 0$ and $y = \theta$, we obtain

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

  Hence

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

Properties of e^z

Theorem

Suppose z, w are complex numbers and n is a positive integer. Then

1. $e^z e^w = e^{z+w}$
2. $\dfrac{e^z}{e^w} = e^{z-w}$
3. $(e^z)^n = e^{nz}$
4. $|e^z| = e^x = e^{\mathrm{Re}\, z}$
5. e^z *is periodic with the imaginary period* $2\pi i$
6. *If k is an integer, then*
   6.1 $e^z = 1$ *if and only if* $z = 2k\pi i$.
   6.2 $e^z = e^w$ *if and only if* $z = w + 2k\pi i$.
7. *If $g(z)$ is analytic, then* $\frac{d}{dz} e^{g(z)} = e^{g(z)} g'(z)$.

Complex Trigonometric Functions

From the Euler's formula,

$$e^{ix} = \cos x + i \sin x, \quad e^{-ix} = \cos x - i \sin x,$$

where x is real. Adding and subtracting these two equations:

$$e^{ix} + e^{-ix} = 2 \cos x, \quad e^{ix} - e^{-ix} = 2i \sin x$$

which are equivalent to

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

This suggests

Definition

If $z = x + iy$, then the complex cosinus and sinus functions of a complex variable z are defined respectively by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Properties of Complex Sine and Cosine Functions

1. $\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$
2. $\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$
3. The inequalities $|\sin x| \leq 1$ and $|\cos x| \leq 1$ are no longer true for $\sin z$ and $\cos z$.
4. The equation $\sin z = 0$ has solutions on the complex plane only at $z = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$.
5. The equation $\cos z = 0$ has solutions on the complex plane only at $z = (2n+1)\pi/2$, $n = 0, \pm 1, \pm 2, \ldots$.
6. $\sin(-z) = -\sin z \quad \cos(-z) = \cos z$
7. $\sin^2 z + \cos^2 z = 1$
8. $\sin(z \pm w) = \sin z \cos w \pm \cos z \sin w$
9. $\cos(z \pm w) = \cos z \cos w \mp \sin z \sin w$
10. $\sin(2z) = 2 \sin z \cos w, \quad \cos(2z) = \cos^2 z - \sin^2 z$
11. $\sin(z + \frac{\pi}{2}) = \cos z$
12. $\sin(\overline{z}) = \overline{\sin z}, \quad \cos(\overline{z}) = \overline{\cos z}$
13. $\frac{d}{dz} \sin z = \cos z, \quad \frac{d}{dz} \cos z = -\sin z$

The following other trigometric functions of a complex variable are defined in the same way as in the real case:

$$\tan z = \frac{\sin z}{\cos z}, \cot z = \frac{\cos z}{\sin z} = \frac{1}{\tan z}, \csc z = \frac{1}{\sin z}, \sec z = \frac{1}{\cos z}.$$

Based on our calculus knowledge, we can immediately conclude that

$$\frac{d}{dz}\tan z = \sec^2 z,$$
$$\frac{d}{dz}\cot z = -\csc^2 z,$$
$$\frac{d}{dz}\csc z = -\csc z \cot z,$$
$$\frac{d}{dz}\sec z = \sec z \tan z.$$

Complex Hyperbolic Functions

The real hyperbolic functions are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2},$$
$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

This suggests that we define the complex hyperbolic functions as

$$\sinh z = \frac{e^z - e^{-z}}{2}, \qquad (1)$$
$$\cosh z = \frac{e^z + e^{-z}}{2}. \qquad (2)$$

Properties of Complex Sinh and Cosh Functions

1. The functions $\cosh z$ and $\sinh z$ are both entire functions.
2. $\sinh z = \sinh x \cos y + i \cosh x \sin y$
3. $\cosh z = \cosh x \cos y + i \sinh x \sin y$.
4. $\frac{d}{dz} \cosh z = \sinh z, \quad \frac{d}{dz} \sinh z = \cosh z$.
5. $\sinh(iz) = i \sin z, \quad \cosh(iz) = \cos z$
6. $\cosh^2 z - \sinh^2 z = 1$
7. $\sinh(z \pm w) = \sinh z \cosh w \pm \cosh z \sinh w$
8. $\cosh(z \pm w) = \cosh z \cosh w \pm \sinh z \sinh w$
9. $\sinh(2z) = 2 \sinh z \cosh w$
10. $\cosh(2z) = \cosh^2 z + \sinh^2 z$
11. The functions $\sinh z$ and $\cosh z$ are each periodic with imaginary period $2\pi i$, a property not found in the real case.
12. The solutions of $\sinh z = 0$ lie on the imaginary axis.
13. All roots of $\cosh z = 0$ also lie on the imaginary axis, i.e.,

$$z = \pm \frac{2n+1}{2} \pi i, \quad n = 0, 1, 2, \ldots.$$

The other complex hyperbolic functions are defined in the same manner as in the real case:

$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{\cosh z}{\sinh z} = \frac{1}{\tanh z},$$
$$\operatorname{csch} z = \frac{1}{\sinh z}, \qquad \operatorname{sech} z = \frac{1}{\cosh z}.$$

Thus it can be shown that

$$\begin{aligned}
(\tanh z)' &= \operatorname{sech}^2 z,\\
(\coth z)' &= -\operatorname{csch}^2 z,\\
(\operatorname{csch} z)' &= -\operatorname{csch} z \coth z,\\
(\operatorname{sech} z)' &= -\operatorname{sech} z \tanh z.
\end{aligned}$$

Complex Logarithmic Function

In calculus,

$$x = e^y \Leftrightarrow y = \ln x \Rightarrow e^{\ln x} = x.$$

This suggests we define the complex logarithm function $\ln z$ such that it satisfies

$$e^{\ln z} = z.$$

Definition

For $z \neq 0$, define

$$\ln z = \ln |z| + i \arg z.$$

WHY?

- $\ln |z|$ may be computed with the help of a calculator.
- Since $\arg z$ is multiple valued, then so is the function $\ln z$.

- In calculus, $\ln 1 = 0$. In complex variables, $\ln 1$ has infinitely many values.
- In calculus, $\ln(-1)$ is undefined since $\ln x$ is valid only for $x > 0$. In complex variables, $\ln(-1)$ is meaningful.
- In calculus, the formula

$$\ln(xy) = \ln x + \ln y$$

is valid with the restriction that x and y are positive reals. If w and z are complex, can we still have

$$\ln(zw) = \ln z + \ln w?$$

If it is so, what are the restrictions on z and w?

The Laws of Complex Logarithm

The following relations hold for certain specified values of the logarithms:

$$\begin{aligned} \ln(zw) &= \ln z + \ln w \\ \ln(z/w) &= \ln z - \ln w \\ \ln z^{n/m} &= \frac{n}{m} \ln z \\ \ln e^{z} &= z \end{aligned}$$

Principal Value Logarithm

- The function ln z can be made single valued by suitably restricting the range of values for arg z.
- Recall that the value of arg z can be made unique by restricting it to the interval $-\pi < \theta \leq \pi$. This unique value is called the principal argument of z and is denoted by $\mathrm{Arg}\, z$.
- This suggests defining the *principal value* of ln z, denoted by $\mathrm{Ln}\, z$, as the value of ln z that employs the principal argument of z, i.e.,

$$\mathrm{Ln}\, z = \ln |z| + i\mathrm{Arg}\, z.$$

Threfore the function $\mathrm{Ln}\, z$ always has a unique value.

Continuity of $\mathrm{Ln}\, z = \ln|z| + i\mathrm{Arg}\, z$

- $\mathrm{Ln}\, z$ is a combination of two functions $\ln|z|$ and $\mathrm{Arg}\, z$.
- The function $\ln|z|$ is continuous on the entire complex plane except at the point $z = 0$.
- $\mathrm{Arg}\, z$ satisfies $-\pi < \mathrm{Arg}\, z \leq \pi$. $\mathrm{Arg}\, z$ is not continuous at $z = 0$ because it is undefined there. $\mathrm{Arg}\, z$ is also not continuous along the negative $\mathrm{Re}\, z$-axis. (WHY?)
- Conclusion, the function $\mathrm{Ln}\, z$ is continuous on a region R consisting of the entire complex plane with the negative $\mathrm{Re}\, z$-axis removed. The line $\mathrm{Re}\, z \leq 0$ is called the **branch cut** for $\mathrm{Ln}\, z$.
- In other words, the function $\mathrm{Ln}\, z$ fails to be continuous at points z such that $\mathrm{Re}\, z \leq 0$ and $\mathrm{Im}\, z = 0$.
- In general, the function $\mathrm{Ln}\,(f(z))$ fails to be continuous at points z such that

$$\mathrm{Re}\, f(z) \leq 0 \quad \text{and} \quad \mathrm{Im}\, f(z) = 0.$$

Derivative of $\mathrm{Ln}\, z$

In calculus, it is proven that

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

for $x > 0$. The following theorem shows that the derivative formula also holds for $\mathrm{Ln}\, z$.

Theorem

Let R denote the domain consisting of the complex plane with the branch cut removed. Then $\mathrm{Ln}\, z$ is analytic on R, and

$$\frac{d}{dz}\mathrm{Ln}\, z = \frac{1}{z}.$$

The domains of continuity and analyticity of $\mathrm{Ln}\, z$ are the same, i.e., $\mathrm{Ln}\, z$ is not analytic at points z such that

$$\mathrm{Re}\, z \leq 0 \quad \text{and} \quad \mathrm{Im}\, z = 0.$$