Chap 2: Geometry of Complex Numbers

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Chap 2: Geometry of Complex Numbers

Outline:

- Complex plane
- Polar form of complex numbers
- De Moivre's formula
- Roots of complex numbers

Geometry of Complex Numbers

The complex plane:

- A simple way of visualizing a complex number $z = x + iy$ is by regarding it to be a point with ordered pair (x, y) in the Cartesian xy-plane.
- The x-axis will be refered the **real axis** and the y-axis will be refered the **imaginary axis**. Such a plane is known as a **complex plane** or z**-plane**.
- The geometric representation of complex numbers is due to the German mathematician Karl Friedrich Gauss (1777-1855), Jean Robert Argand (1768-1822, French), and Caspar Wessel (1745-1818, Norwegian).

Complex Conjugate of a Complex Number

- The **complex conjugate** of a complex number $z = x + iy$ is denoted by $\overline{z} = x - iy$.
- The point $\overline{z}$ is the reflection of the point z with respect to the real axis.
- The distance OP gives the value of $|z|$, while the distance OQ gives the value of $|\overline{z}|$.
- The distance between z and w is measured by $|z - w|$.

Polar Form of a Complex Number

- In calculus, the Cartesian coordinates (x, y) may be expressed in terms of the polar coordinates (r, θ).
- r measures the distance of the point from the origin
- θ measures the angle of the straight line, joining the point to the origin O, with respect to the real axis.

- Relationship between (x, y) and (r, θ):

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = \sqrt{x^2 + y^2}, \quad \tan\theta = y/x.$$

- **Polar representation** or **polar form** of the complex number $z = x + iy$:

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta).$$

- The term r is called the **modulus** of z.
- The term θ is called the **argument** or **phase** of z, and is denoted by $\arg z$.
- The value of $\arg z$ is however not unique.
- The value of $\arg z$ can be made unique by restricting it to the interval $-\pi < \theta \leq \pi$. This unique value is called the **principal argument** of z and is denoted by $\mathrm{Arg}\, z$.
- As for the case $z = 0$, its modulus is zero but its argument is left undefined.

Multiplication in Polar Forms

Suppose we are given two complex numbers

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1), \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

Then

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

Hence,

$$|z_1 z_2| = r_1 r_2,$$
$$|z_1/z_2| = r_1/r_2,$$
$$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2),$$
$$\arg(z_1/z_2) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$$

De Moivre's formula

If $z = r(\cos\theta + i\sin\theta)$, n any integers, then

$$z^n = r^n(\cos n\theta + i\sin n\theta),$$

If $r = 1$, the equation becomes

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta = e^{in\theta}$$

known as **De Moivre's formula**, after the French mathematician Abraham De Moivre (1667-1754).

- De Moivre's formula is very usefull when converting a complex number raised to a big power into the form $a + ib$.
- De Moivre's formula is also helpful in deriving several trigonometric identities involving multiple angles:

$$\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta, \quad \sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

Roots of Complex Numbers

Definition

Suppose n is a positive integer. The nth root of a complex number z_0, written as $z_0^{1/n}$, is all the complex numbers z satisfying

$$z^n = z_0.$$

- The nth roots of a complex number z, denoted by $z^{1/n}$, for n positive integers.
- If $z = r(\cos\theta + i\sin\theta)$, then, based on De Moivre's formula, it can be shown that

$$z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right],$$

with $k = 0, 1, 2, \ldots, n-1$.

Roots of Complex Numbers (Cont.)

More generally, if $z = r(\cos\theta + i\sin\theta)$, $m = \pm 1, \pm 2, \ldots$, $n = 1, 2, 3, \ldots$, and m and n have no common factors, then

$$z^{m/n} = r^{m/n}\left[\cos\left(\frac{m\theta + 2km\pi}{n}\right) + i\sin\left(\frac{m\theta + 2km\pi}{n}\right)\right],$$

with $k = 0, 1, 2, \ldots, n-1$.

Sets of Complex Numbers

In calculus, functions of a real variable are usually defined over intervals. Functions of a complex variable however are defined over subsets of complex plane. This section presents some important sets of complex numbers, which will assists us in our study of functions of a complex variable.

The set $\mathrm{Im}\, z = 0$ is the real axis, while the set $\mathrm{Re}\, z = 0$ is the imaginary axis.

Definition (Half planes)

A set of complex numbers z such that $\mathrm{Im}\,(z) > 0$ is called an **upper-half plane**. A set of complex numbers z such that $\mathrm{Im}\,(z) < 0$ is called a **lower-half plane**. The **right-half plane** is described by $\mathrm{Re}\, z > 0$, while the **left-half plane** is described by $\mathrm{Re}\, z < 0$.

Circle, Disk

Definition (Circle, Disk)

The set of complex numbers z such that

$$|z - z_0| = r.$$

is a **circle** centered at z_0 with radius r. The set

$$|z - z_0| < r$$

is called a **disk.**

We defined a **unit circle** by $|z| = 1$, and a **unit disk** by $|z| < 1$.

Neighborhood, Interior points, Open sets

Definition (Neighborhood)

The set of all points z inside a circle of radius σ and centered at the point z_0, i.e., $|z - z_0| < \sigma$, is called a **neighborhood** or σ**-neighborhood**of z_0.

Note that a σ-neighborhood of z_0 does not include the points on the circle $|z - z_0| = \sigma$.

Definition (Interior points, Open sets)

A point z_0 is called an **interior point** of a set D if there exists at least one neighborhood of z_0 that is wholly contained in D. The collection of all interior points of D is called the **interior** of D. If every point of D is an interior point, then D is called **open**.

Exterior Points

Definition (Exterior points)

A point z_0 is called an **exterior point** of a set D if there is a neighborhood of z_0 that contains no points of D. The collection of all exterior points of D is called the **exterior** of D.

Boundary points, Closed sets

Definition (Boundary points, Closed sets)

A **boundary point** of a set D is a point which is neither an interior nor an exterior point of D. The set of all boundary points of D is called the **boundary** of D. A set is **closed** if it contains all its boundary points.

Thus a boundary point of a set D has the property that its neighborhood contains points in D as well as points not in D. For example, the set of points such that

$$0 < |z - 2| \leq 1,$$

has the set of points on the circle $|z - 2| = 1$ and the point $z = 2$ as its boundaries. Such a set is also called a **punctured disk** or **deleted neighborhood** of $z = 1$.

Bounded and Unbounded Sets

Definition (Bounded and unbounded sets)

A set is D called **bounded** if there is a constant R such that $|z| < R$ for every point z in D; otherwise, D is **unbounded.** A set that is both closed and bounded is sometimes called **compact.**

Connected sets

Definition (Connected sets)

A set S is called **connected** if every pair of points in D can be joined by a successive line segments (polygonal path) that lies entirely in D; otherwise, the set is **not connected.**

Roughly speaking, a connected set consists of a single piece. A connected set may contains countable number of holes. A connected set which has no holes is said to be **simply connected**; otherwise, it is **multiply connected**.

Domains, Regions

Definition (Domains, Regions)

An open connected set is called a **domain**, while a **region** is a domain together with some, none, or all of its boundary points.

Point at Infinity, i.e, $z = \infty$

- On the real number line, there are two directions that give rise to $\pm\infty$.
- In the complex number system, however, there is only one point of infinity $z = \infty$ which can be attained in several directions.
- No positive or negative signs are assigned to complex infinity.
- The argument of complex infinity is also left undefined.