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UNIVERSITI TEKNOLOGI MALAYSIA

Chap 1: Complex Numbers

Ali H. M. Murid

Department of Mathematical Sciences,
Faculty of Science, Universiti Teknologi Malaysia,
81310 UTM Johor Bahru, Malaysia
alihassan@utm.my

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Chap 1: Complex Numbers

Outline:

- Brief History of Complex Numbers
- Arithmetic operations on complex numbers
- Conjugate and absolute value of complex numbers
- Inequalities
- Complex plane
- Polar form of complex numbers
- De Moivre's formula
- Roots of complex numbers

Brief history of complex numbers

- Complex numbers were invented in the 16th century when mathematicians were looking for solutions to the quadratic and cubic equations.
- The simple quadratic equation

$$x^2 + 1 = 0$$

has no real number solution.

- We may however write the solutions as

$$x = \pm\sqrt{-1}.$$

But $\sqrt{-1}$ is not a real number.

- In general, the ancient Greeks/Egyptians in the 3rd century knew how to solve

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- Mainly interested with +ve roots. Rejected -ve numbers (no physical interpretation).
- Historically, the understanding of $\sqrt{-1}$ came not from the quadratics but rather from the cubic equations.

Scipione del Ferro (1465-1526, Italian)

Discovered that the depressed cubic

$$x^3 + px = q,$$

has a solution given by the formula

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

For example,

$$x^3 + 6x = 20, \quad p = 6, q = 20,$$

implies

$$x = \sqrt[3]{10 + \sqrt{168}} - \sqrt[3]{-10 + \sqrt{168}} = 2.$$

Girolamo Cardano (1501-1576, Italian)

Discovered a method for solving the general cubic

$$x^3 + ax^2 + bx + c = 0.$$

Substitution

$$x = y - a/3 \Rightarrow y^3 + py = q,$$

with

$$p = b - \frac{1}{3}a^2, \quad q = -\frac{2}{27}a^3 + \frac{1}{3}ab - c.$$

What really puzzled Cardano:
The existence of $\sqrt{-\mathrm{ve}}$ occuring in Cardano's formula that had clearly real solutions.

For example,

$$x^3 - 15x = 4 \Rightarrow x = 4, \quad x = -2 \pm \sqrt{3}.$$

All reals.
Cardano's formula gives

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{-2 + \sqrt{-121}}.$$

This must be a real number.

Rafael Bombelli (1526-1572, Italian, Cardano's follower)

Assume

$$\sqrt[3]{2+\sqrt{-121}} = A + B\sqrt{-1} \Rightarrow (A + B\sqrt{-1})^3 = 2 + \sqrt{-121},$$
$$\sqrt[3]{2-\sqrt{-121}} = A - B\sqrt{-1} \Rightarrow (A - B\sqrt{-1})^3 = 2 - \sqrt{-121}$$

which implies $A = 2, B = 1$. Therefore

$$\sqrt[3]{2+\sqrt{-121}} = A + B\sqrt{-1} = 2 + \sqrt{-1},$$
$$\sqrt[3]{2-\sqrt{-121}} = A - B\sqrt{-1} = 2 - \sqrt{-1},$$

which yields

$$x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{-2+\sqrt{-121}} = (2+\sqrt{-1}) + (2-\sqrt{-1}) = 4.$$

- From Bombelli's work, it became clear that manipulations of $\sqrt{-1}$ using the ordinary rule of arithmetic leads to perfectly correct results.
- In 1777 Leonhard Euler (1707−1783, Switzerland) introduced the notation i to represent $\sqrt{-1}$ with the basic property $i^2 = -1$. He wrote $a + ib$ to represent a complex number.
- The symbol i is called an *imaginary number* because it was shrouded with mysteries.

- Caspar Wessel (1745–1818, Norway) in 1797 and Carl Friedrich Gauss (1777–1855, German) in 1799 gave a simple geometric representation for the complex number $a + ib$.
- In 1833, Sir William Rowan Hamilton (1805–1865, Ireland) provided a formal algebraic presentation for the complex number system.

Concept of complex numbers

Definition (Complex Number)

A number z is called a **complex number** if it can be written in the form

$$z = x + iy,$$

where x and y are two real numbers and i is an imaginary unit with the property $i^2 = -1$.

The number x is called the **real part** of z, and is written

$$x = \operatorname{Re} z$$

while the number y is called the **imaginary part** of z, and is written

$$y = \operatorname{Im} z.$$

Arithmetic operations on complex numbers

Equality of two complex numbers:
Suppose that we are given two complex numbers

$$z_1 = x_1 + iy_1, \qquad z_2 = x_2 + iy_2.$$

The complex number z_1 is **equal** to the complex number z_2 if and only if

$$x_1 = x_2, \qquad y_1 = y_2.$$

Addition and subtraction of complex numbers:
Assume that we are given two complex numbers

$$z_1 = x_1 + iy_1, \qquad z_2 = x_2 + iy_2.$$

Adding z_1 and z_2 will yield another complex number:

$$\begin{aligned} z_1 + z_2 &= x_1 + iy_1 + x_2 + iy_2 \\ &= x_1 + x_2 + iy_1 + iy_2 \\ &= x_1 + x_2 + i(y_1 + y_2). \end{aligned}$$

Subtraction of two complex numbers z_1 and z_2 will also yield another complex number:

$$\begin{aligned} z_1 - z_2 &= x_1 + iy_1 - (x_2 + iy_2) \\ &= x_1 + iy_1 - x_2 - iy_2 \\ &= x_1 - x_2 + iy_1 - iy_2 \\ &= x_1 - x_2 + i(y_1 - y_2). \end{aligned}$$

Thus in combining complex numbers, we combine the real parts together and the imaginary parts together.

Multiplication and division of complex numbers:
The repetitive multiplication of the imaginary unit is rather easy. The imaginary unit i has the property

$$i^2 = -1.$$

This leads to

$$\begin{aligned} i^3 &= (i^2)i = (-1)i = -i \\ i^4 &= (i^2)^2 = (-1)^2 = 1 \\ i^{2n} &= (i^2)^n = (-1)^n, \quad n = 1, 2, 3, \ldots \\ i^{2n+1} &= (i^2)^n i = (-1)^n i, \quad n = 0, 1, 2, \ldots \end{aligned}$$

In general, the multiplication of two complex numbers obeys the same rule as multiplying two real numbers $a + b$ with $c + d$:

$$(a + b)(c + d) = ac + ad + bc + bd.$$

Any occurence of i^2 is replaced by -1. Thus if we are given two complex numbers

$$z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2,$$

then,

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + ix_1 y_2 + iy_1 x_2 + i^2 y_1 y_2 \\ &= x_1 x_2 + ix_1 y_2 + iy_1 x_2 - y_1 y_2 = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1) \end{aligned}$$

For the evaluation of the division involving two complex numbers z_1 and z_2, i.e.,

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}, \quad z_2 \neq 0, \tag{1}$$

we proceed as follows:

$$\begin{aligned}\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} &= \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} \\ &= \frac{x_1x_2 + y_1y_2 + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \\ &= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}.\end{aligned}$$

Conjugate and absolute value of complex numbers:

Definition (Complex Conjugate)

The **complex conjugate** of the complex number $z = x + iy$, denoted by $\overline{z}$, is given by

$$\overline{z} = x - iy.$$

In general, from the definition above, $\overline{z} = z$ if and only if z is a real number, and if $z = x + iy$, we have

$$x = \frac{z + \overline{z}}{2}, \quad y = \frac{z - \overline{z}}{2i}.$$

Further results on complex conjugation are listed below:

$$
\begin{aligned}
\overline{\overline{z}} &= z \\
\overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2} \\
\overline{z_1 - z_2} &= \overline{z_1} - \overline{z_2} \\
\overline{z_1 z_2} &= \overline{z_1}\,\overline{z_2} \\
\overline{\left(\frac{z_1}{z_2}\right)} &= \frac{\overline{z_1}}{\overline{z_2}}, \qquad (z_2 \neq 0)
\end{aligned}
$$

Definition (Absolute Value or Modulus)

The **absolute value** or **modulus** of a complex number $z = x + iy$, denoted by $|z|$, is a real number given by

$$|z| = \sqrt{x^2 + y^2}.$$

If $z = x + iy$, it is readily observed that

$$z\overline{z} = (x + iy)(x - iy) = x^2 + y^2.$$

From the definition above, $|z|^2 = x^2 + y^2$. Hence the above equation becomes

$$z\overline{z} = |z|^2.$$

Further properties of complex numbers related to modulus are listed below:

$$
\begin{aligned}
|\overline{z}| &= |z| \\
|z| &= 0 \quad \text{iff} \quad z = 0 \\
|z_1 z_2| &= |z_1||z_2| \\
\left|\frac{z_1}{z_2}\right| &= \frac{|z_1|}{|z_2|}, \quad \text{if} \quad z_2 \neq 0.
\end{aligned}
$$

Inequalities:

It is impossible to arrange the complex numbers either in decreasing or increasing order like the real numbers. Listed below are some inequalities with respect to complex numbers:

(a) $-|z| \leq \mathrm{Re}\,(z) \leq |z|$

(b) $-|z| \leq \mathrm{Im}\,(z) \leq |z|$

(c) $|z_1 + z_2| \leq |z_1| + |z_2|$

(d) $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$

(e) $|z_1| - |z_2| \leq |z_1 - z_2|$

(f) $|z_1 - z_2| \geq ||z_1| - |z_2||$

(g) $|w_1 z_1 + w_2 z_2 + \cdots + w_n z_n|^2 \leq (|w_1|^2 + |w_2|^2 + \cdots + |w_n|^2)(|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)$

Inequality (c) is known as the **triangle inequality**, while inequality (g) is called the **Cauchy's inequality**.