SEL4223 Digital Signal Processing

IIR Filter Design

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Digital Filter Design

- Filtering is a process of changing signal’s spectral content. The change is usually to **attenuate a range of frequencies** in the signal while allowing the other frequencies to pass through.

- Below are how z-transform and DTFT are used to design the filter

<table>
<thead>
<tr>
<th>Filter</th>
<th>Z-transform</th>
<th>DTFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>IIR</td>
<td>• To convert an analog filter to the digital filter</td>
<td>• To analyze the spectral response</td>
</tr>
<tr>
<td>(bilinear transformation)</td>
<td>• To obtain difference equation</td>
<td></td>
</tr>
<tr>
<td>FIR</td>
<td>• <em>Not used. Difference equation can be obtained in time-domain by convolution</em></td>
<td>• To analyze the spectral response</td>
</tr>
<tr>
<td>(windowing)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Digital Filter Design (cont.)

- Shown in the previous table are only for bilinear transformation and windowing techniques. There are many other techniques in designing both IIR and FIR filters. Different technique will use the z-transform and DTFT (or DFT) differently.
Ideal Filter

- Cut-off frequency ($\omega_c$) is the only parameter considered.

**Lowpass filter:**

\[
|H(\omega)|_{LP} = \begin{cases} 
1 & \text{for } \omega \leq \omega_c \\
0 & \text{for } \omega_c < \omega < \pi 
\end{cases}
\]

**Highpass filter:**

\[
|H(\omega)|_{HP} = \begin{cases} 
0 & \text{for } \omega < \omega_c \\
1 & \text{for } \omega_c \leq \omega \leq \pi 
\end{cases} = 1 - |H(\omega)|_{LP}
\]
Ideal Filter (cont.)

Bandpass filter:

\[
|H(\omega)|_{HP} = \begin{cases} 
0 & \text{for } \omega < \omega_{c1} \\
1 & \text{for } \omega_{c1} \leq \omega \leq \omega_{c2} \\
0 & \text{for } \omega_{c2} < \omega \leq \pi 
\end{cases}
\]

\[= |H(\omega)|_{LP(\omega_{c2})} - |H(\omega)|_{LP(\omega_{c1})}\]

Bandstop filter:

\[
|H(\omega)|_{HP} = \begin{cases} 
1 & \text{for } \omega \leq \omega_{c1} \\
0 & \text{for } \omega_{c1} < \omega < \omega_{c2} \\
1 & \text{for } \omega_{c2} \leq \omega \leq \pi 
\end{cases}
\]

\[= 1 - |H(\omega)|_{LP(\omega_{c2})} + |H(\omega)|_{LP(\omega_{c1})}\]
Non-ideal Filter

- Filter characteristic below must be considered:

\( \omega_c \) - Cutoff frequency
\( \omega_p \) - Passband edge frequency
\( \omega_s \) - Stopband edge frequency
\( \delta_p \) - Passband ripple
\( \delta_s \) - Stopband ripple
\( N \) - Filter order
\( \Delta \omega \) - Transition bandwidth
Non-ideal Filter (cont.)

- Passband ripple: $1 + \delta_1$
- Stopband: $\delta_2$
- Transition band: $\omega_p$ to $\omega_s$

The graph illustrates the frequency response of a non-ideal filter, showing the magnitude of the frequency response $|H(\omega)|$ with different regions labeled for passband, transition band, and stopband.
IIR Filter Design

• There are two common techniques used in designing the IIR filter
  – Impulse Invariance
  – Bilinear Transformation

• Basically, both techniques are implemented by converting system function of continuous-time filter \( H(s) \) to the discrete-time system function \( H(z) \). In other words, they map all poles in \( s \)-plane onto \( z \)-plane.
Bilinear Transformation

- In Bilinear transformation technique, relationship between the s-plane and z-plane is shown below where $c = \frac{2}{T_s}$ and $T_s$ is the time sampling.

\[ s = c \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right) \]

- Then, the relationship between the continuous-time frequency ($\Omega$) and the discrete-time frequency ($\omega$) is

\[ \Omega = c \cdot \tan \left( \frac{\omega}{2} \right) \quad \text{and} \quad \omega = 2 \tan^{-1} \left( \frac{\Omega}{c} \right) \]
Filter Design Procedure

1. Determine filter characteristic \((\delta_p, \delta_s, \omega_p, \omega_s, \omega_c, N)\):

   When designing filter, not all filter characteristics must be determine. Below are 3 ways of specifying the filter characteristics.

   I. Specify \(\delta_p, \delta_s, \omega_p\) and \(\omega_s\)

   II. Specify \(\omega_c, N\)

   III. Specify \(\omega_c, \omega_s\) and \(\delta_s\) or \(\omega_c, \omega_p\) and \(\delta_p\)

2. Find system function of the continuous-time filter, \(H(s)\):

   For Butterworth filter, need to find \(\Omega_c\) and \(N\).

3. Transform the continuous-time filter, \(H(s)\) to the discrete-time filter, \(H(z)\)

4. Obtain the time-domain representation of the discrete-time filter for implementation:

   Either as an impulse response or as a difference equation.
Butterworth filter

• In this class, the filter design will be based only on Butterworth filter, which is one of the well known continuous-time filter. Another example of well known continuous-time filter is Chebyshev filter.

• The magnitude squared spectrum of continuous Butterworth filter is define as:

\[
|H(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}
\]
Butterworth filter (cont.)

• From there, it follows that

\[ H(s)H(-s) = \frac{1}{1 + \left(\frac{-s^2}{\Omega_c^2}\right)^N} \]

• Based on the previous equation, it shows that Butterworth filter is an IIR filter as it contains poles at \( s \neq 0 \). From the equation, it also shows that Butterworth filter contains only poles and no zeros on the s-plane.
Butterworth filter (cont.)

- The poles of the Butterworth filter can be determined as follow:

\[ s_k = \Omega_c e^{j(2k+N+1)\pi/2N}, \quad k = 0, 1, \ldots, N - 1 \]

- Total number of the poles will be similar to \( N \) (filter order) where all poles are positions at \( \sigma < 0 \) on the s-plane. This is to ensure the causality and stability of the filter.

- The following figures are examples of the poles position on the s-plane with \( \Omega_c = 1 \).
Butterworth filter (cont.)
Butterworth filter (cont.)

• Then, the system function of the Butterworth filter is

\[ H(s) = \frac{\Omega_c^N}{\prod_{k=0}^{N-1} (s - s_k)} \]

\[ s_k = e^{j(2k+N+1)\pi/2N}, \quad k = 0, 1, \ldots, N - 1 \]

• To simplify the system function, always set \( \Omega_c = 1 \). Thus, the system function becomes

\[ H(s) = \frac{1}{\prod_{k=0}^{N-1} (s - s_k)} \]
Butterworth filter (cont.)

- Below is table showing the system function for several filter order when $\Omega_c = 1$.

<table>
<thead>
<tr>
<th>N</th>
<th>$H(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{s + 1}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{s^2 + 1.4142s + 1}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{(s + 1)(s^2 + s + 1)}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{1}{(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)}$</td>
</tr>
</tbody>
</table>
Example 1

• Design a digital lowpass filter based on 2\textsuperscript{nd} order Butterworth filter where cutoff frequency of the filter is $\omega_c = 0.5\pi \text{ rad}$

Solution:

Step 1: Specify filter characteristics. Use given $\omega_c = 0.5\pi$ and $N = 2$.

Step 2: Find system function of the continuous filter by setting $\Omega_c = 1$, system function for 2\textsuperscript{nd} order Butterworth filter is

$$H(s) = \frac{1}{s^2 + 1.4142s + 1}$$
Step 3: Transform $H(s)$ to $H(z)$

\[ s = c \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right) \]

Need to find $c$ value. It can be computed based on given $\omega_c$ and $\Omega_c = 1$ using equation below

\[ \Omega_c = c \cdot \tan \left( \frac{\omega_c}{2} \right) \]

\[ c = \frac{1}{\tan(0.25\pi)} = 1 \]
Then, the discrete-time system function is

\[ H(z) = \frac{1}{\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)^2 + 1.4142\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) + 1} \]

\[ = \frac{(1 + z^{-1})^2}{(1 - z^{-1})^2 + 1.4142(1 - z^{-1})(1 + z^{-1}) + (1 + z^{-1})^2} \]

\[ = \frac{(1 + z^{-1})^2}{3.4142 + 0.5858z^{-1}} \]

\[ = 0.2929 \frac{1 + 2z^{-1} + z^{-2}}{1 + 0.1864z^{-2}} \]
Step 4: Obtain time-domain representation. Here we use difference equation.

\[ y[n] = 0.2929(x[n] + 2x[n - 1] + x[n - 2]) - 0.1864y[n - 2] \]

- In order to see the shape of the filter, obtain and plot \(|H(\omega)|\). For this example, the plot is shown below. Also shown is the magnitude dB plot and poles and zero plot for the filter.
Example 1 (cont.)

\[ |H(\omega)| \]

\[ \omega (\pi \text{ rad}) \]

\[ |H'(\omega)|_{db} \]

\[ \omega (\pi \text{ rad}) \]
Example 1 (cont.)
Example 2

• Design an IIR lowpass filter based on Butterworth filter with the following filter characteristics.

\[ \delta_s = \delta_p = 0.1 \]
\[ \omega_p = 0.2\pi \]
\[ \omega_s = 0.4\pi \]

Solution:

Step 1: Specify filter characteristics. As given in the question the filter characteristics are shown in the following figure.
Example 2 (cont.)

**Step 2:** Find system function of the continuous filter

To obtain the system function, set $\Omega_c = 1$ and find $N$
Example 2 (cont.)

- From the filter characteristics, $|H(j\Omega)|^2$ at $\Omega_p$ and $\Omega_s$ can be identified, which are $0.9^2$ and $0.1^2$ respectively. Based on this information, $N$ can be computed as follows where generally, the magnitude squared spectrum of Butterworth filter is set as $\Omega_c = 1$

$$|H(j\Omega)|^2 = \frac{1}{1 + \Omega^{2N}}$$

- In order to compute $N$, evaluate the magnitude squared spectrum at $\Omega_p$ and $\Omega_s$. Based on bilinear transformation;

$$\Omega = c \cdot \tan \left( \frac{\omega}{2} \right)$$
Example 2 (cont.)

• \( \Omega_p = c \cdot \tan \left( \frac{\omega_p}{2} \right) = c \cdot \tan \left( \frac{0.2\pi}{2} \right) = 0.3249c \)

• \( \Omega_s = c \cdot \tan \left( \frac{\omega_s}{2} \right) = c \cdot \tan \left( \frac{0.4\pi}{2} \right) = 0.7265c \)

• Evaluating magnitude squared spectrum at \( \Omega_p \) and \( \Omega_s \) gets to

\[
\frac{1}{1 + (0.3249c)^{2N}} = 0.9^2 \tag{1}
\]

\[
\frac{1}{1 + (0.7265c)^{2N}} = 0.1^2 \tag{2}
\]
Example 2 (cont.)

• By manipulating and rearranging the two equations, it can be shown that

\[ N = \frac{1}{2} \left( \log \left( \frac{|H(\Omega_S)|^2 \cdot (1 - |H(\Omega_p)|^2)}{|H(\Omega_p)|^2 \cdot (1 - |H(\Omega_S)|^2)} \right) \right) \]

\[ = 3.7569 \approx 4 \]

• Because \( N \) must be an integer number, value from the computation is round toward infinity to ensure the filter characteristics specified in step 1 is hold.
Example 2 (cont.)

Finally, with $\Omega_c = 1$ and $N = 4$, the system function of the continuous-time filter is

$$H(s) = \frac{1}{(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)}$$

Step 3: Transform $H(s)$ to $H(z)$

$$s = c \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right)$$

Need to find $c$ value. It can be computed based on equation (1) with $N = 4$. The results is $c = 2.5676$
Based on the $c$ value, the discrete-time system function is

$$H(z) = \frac{1}{\left(c^2 \left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.7654c\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 1\right)\left(c^2 \left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 1.8478c\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 1\right)}$$

$$= \frac{(1+z^{-1})^4}{(9.5578-11.1851z^{-1}+5.6273z^{-2})(12.337-11.1851z^{-1}+2.8481z^{-2})}$$

**Step 4:** Obtain time-domain representation. Do it yourself as an exercise.

- Magnitude spectrum, magnitude dB spectrum and pole-zero plot of the filter are shown below where the cutoff frequency is $\omega_c = 0.24$. $\omega_c$ can be computed using equation
  $$\omega = 2\tan^{-1}\left(\frac{\Omega}{c}\right)$$
Example 2 (cont.)

\[ |H(\omega)| \]

\[ |H(\omega)|_{db} \]

\[ \omega (\pi \text{ rad}) \]
Example 2 (cont.)
As in the Example 2, there are two pairs of poles ($N = 4$). Thus, denominator of $H(z)$ is presented by multiplication of two sets of the 2\textsuperscript{nd} order expressions.

For each pair of poles, the transformation using the bilinear transformation from $H(s)$ to $H(z)$ where $\Omega_c = 1$ can be written as

\[
H(s) = \frac{1}{s^2 + as + 1} \quad \Rightarrow \quad H(z) = \frac{(1 + z^{-1})^2}{b_1 + b_2z^{-1} + b_3z^{-2}}
\]

\[
b_1 = c^2 + ac + 1 \\
b_2 = -2c^2 + 2 \\
b_3 = c^2 - ac + 1
\]
Pair of Poles Solution (cont.)

• When $N$ is odd, there will be one extra poles after pairing all conjugation poles. The transformation of the extra poles from $H(s)$ to $H(z)$ where $\Omega_c = 1$ can be written as

\[
H(s) = \frac{1}{s + 1} \quad \Rightarrow \quad H(z) = \frac{(1 + z^{-1})}{d_1 + d_2 z^{-1}}
\]

\[
d_1 = 1 + c \\
d_2 = 1 - c
\]
Example 3

• Convert $H(s)$ to $H(z)$ for the 4th order Butterworth filter shown below using bilinear transformation. Assume $c = 1$

$$H(s) = \frac{1}{(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)}$$

Solution:

• From bilinear transformation, $H(z)$ can written as

$$H(z) = \frac{(1 + z^{-1})^4}{(b_1 + b_2z^{-1} + b_3z^{-2})(d_1 + d_2z^{-1} + d_3z^{-2})}$$
Example 3 (cont.)

where

\[
\begin{align*}
    b_1 &= 1^2 + 0.7654 + 1 = 2.7654 \\
    d_1 &= 1^2 + 1.8478 + 1 = 3.8478 \\
    b_2 &= -2 + 2 = 0 \\
    d_2 &= -2 + 2 = 0 \\
    b_3 &= 1^2 - 0.7654 + 1 = 1.2346 \\
    d_3 &= 1^2 - 1.8478 + 1 = 0.1522
\end{align*}
\]

Thus,

\[
H(z) = \frac{(1 + z^{-1})^4}{(2.7654 + 1.2346z^{-2})(3.8478 + 0.1522z^{-2})}
\]
Example 4

• Design an IIR Butterworth filter with $\omega_c = 0.5\pi$, $\omega_s = 0.9\pi$ and $\delta_s = 0.01$

Solution:

Step 1: Specify filter characteristics. Use filter characteristics as given in the question.

Step 2: Find system function of the continuous filter by computing $N$ from the magnitude squared equation of the Butterworth filter where

$$\frac{1}{1 + \Omega_s^{2N}} = 0.01^2$$
Example 4 (cont.)

\[ \Omega_s = c \cdot \tan \left( \frac{\omega_s}{2} \right) = c \cdot \tan \left( \frac{0.9\pi}{2} \right) = 6.3138c \]

- \( c \) value can be computed from equation below with \( \Omega_c = 1 \)

\[ \Omega_c = c \cdot \tan \left( \frac{\omega_c}{2} \right) \]

- From there, \( c = 1 \). Thus \( \Omega_s = 6.3138 \) and the magnitude squared equation becomes

\[ \frac{1}{1 + 6.3138^{2N}} = 0.01^2 \]
Example 4 (cont.)

• Rearranging the magnitude squared equation leads to the formulation of $N$ as below

\[
N = \frac{1}{2} \cdot \frac{\log \left( \frac{1}{\delta_s^2} - 1 \right)}{\log \Omega_s} = \frac{1}{2} \left( \frac{4}{0.8} \right) = 2.5 \approx 3
\]

• Finally, with $\Omega_c = 1$ and $N = 3$, the system function of the continuous-time filter is

\[
H(s) = \frac{1}{(s + 1)(s^2 + s + 1)}
\]
Step 3: Transform $H(s)$ to $H(z)$

$$s = c \left( \frac{1-z^{-1}}{1+z^{-1}} \right)$$

Thus,

$$H(z) = \frac{(1+z^{-1})(1+z^{-1})^2}{(a_1+a_2z^{-1})(b_1+b_2z^{-1}+b_3z^{-2})}$$

where

$$a_1 = 1 + c = 2$$

$$a_2 = 1 - c = 0$$
Example 4 (cont.)

\[ b_1 = c^2 + ac + 1 = 3.4142 \]
\[ b_2 = -2c^2 + 2 = 0 \]
\[ b_3 = c^2 - ac + 1 = 0.5858 \]

Finally,

\[
H(z) = \frac{(1 + z^{-1})^3}{2(3.4142 + 0.5858z^{-2})}
\]
\[ = 0.1464 \frac{(1 + z^{-1})^3}{(1 + 0.1716z^{-2})} \]
Step 4: Obtain time-domain representation.

\[ H(z) = 0.1416 \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{1 + 0.1716z^{-2}} \]

\[ y[n] = 0.1416(x[n] + 3x[n - 1] + 3x[n - 2] + x[n - 3]) - 0.1716y[n - 2] \]
Example 4 (cont.)

![Graphs showing magnitude and phase of frequency response](image-url)
Example 4 (cont.)

Graph showing a complex plane with points plotted.
Example 5

- Design an IIR Butterworth filter that will attenuate frequencies component at \( \omega = 0.5\pi \) and \( \omega = 0.9\pi \) in signal \( x[n] \) shown below. Also shown are the signal’s magnitude and phase spectrum.
Example 5 (cont.)

\[ |X(\omega)| \]

\[ \phi X(\omega) \]
Example 5: 2\textsuperscript{nd} order BF

Solution:

- Below are the solution by applying 2\textsuperscript{nd} order and 9\textsuperscript{th} order IIR Butterworth filter to signal $x[n]$ with $\omega_c = 0.3\pi$

- 2\textsuperscript{nd} order Butterworth filter:
Example 5: 2\textsuperscript{nd} order BF
Example 5: 2\textsuperscript{nd} order BF

- Output

\begin{align*}
|H(\omega)| &
\begin{cases}
\begin{array}{c}
\omega(\pi \text{ rad})
\end{array}
\end{cases}
\end{align*}

\begin{align*}
\phi H(\omega) &
\begin{cases}
\begin{array}{c}
\omega(\pi \text{ rad})
\end{array}
\end{cases}
\end{align*}
Example 5: $2^{nd}$ order BF

$y[n]$
Example 5: 9\textsuperscript{th} order BF

9\textsuperscript{th} order Butterworth filter

\begin{align*}
|H(\omega)| \quad \omega(\pi \text{ rad})
\end{align*}

\begin{align*}
\phi H(\omega) \quad \omega(\pi \text{ rad})
\end{align*}
Example 5: 9\textsuperscript{th} order BF
Example 5: 9\textsuperscript{th} order BF

- Output

\begin{align*}
|H(\omega)|
\end{align*}

\begin{align*}
\phi H(\omega)
\end{align*}
Example 5: $9^{th}$ order BF

$$y[n]$$

For $n = 0$ to $100$.
References

